MA59800ANT ANALYTIC THEORY OF FUNCTION FIELDS. PROBLEMS 2

TO BE HANDED IN BY 6PM WEDNESDAY 18TH SEPTEMBER 2024

Key: A-questions are short questions testing basic skill sets; B-questions integrate essential methods of the course; C-questions are more challenging for enthusiasts, with hints available on request.

A1. Let $\mathbb{A} = \mathbb{F}_q[t]$, where $q = p^h$, and suppose that $\pi \in \mathbb{A}$ is a monic irreducible. Let $m \in \mathbb{N}$, and write l for the least positive integer with $p^l \geqslant m$. Show that $\{a^{|\pi|-1} : a \in (\mathbb{A}/\pi^m\mathbb{A})^\times\}$ is a subgroup of $(\mathbb{A}/\pi^m\mathbb{A})^\times$ having the structure of an abelian p-group, all of whose elements have order at most p^l.

A2. Let $\mathbb{A} = \mathbb{F}_q[t]$, where $q = p^h$ with p odd, and suppose that $m \in \mathbb{A}$ is a monic polynomial of positive degree which is not irreducible. Is it possible that $(\mathbb{A}/m\mathbb{A})^\times$ is cyclic? Explain your answer.

B3. (i) Let p_1 and p_2 be two distinct Mersenne primes, say $p_1 = 2^{r_1} - 1$ and $p_2 = 2^{r_2} - 1$. Suppose that π_1 and π_2 are two irreducible polynomials in $\mathbb{A} = \mathbb{F}_2[t]$ having respective degrees r_1 and r_2. Prove that $(\mathbb{A}/\pi_1\pi_2\mathbb{A})^\times$ is cyclic.
(ii) Let n_1 and n_2 be two distinct Mersenne numbers, say $n_1 = 2^{r_1} - 1$ and $n_2 = 2^{r_2} - 1$. Suppose that π_1 and π_2 are two irreducible polynomials in $\mathbb{A} = \mathbb{F}_2[t]$ having respective degrees r_1 and r_2. When is it the case that $(\mathbb{A}/\pi_1\pi_2\mathbb{A})^\times$ is cyclic? Explain your answer.

B4. Define the von Mangoldt function in $\mathbb{A} = \mathbb{F}_q[t]$ by putting

$$\Lambda(u) = \begin{cases} \log|\pi|, & \text{when } u = \pi^r \text{ for some } r \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Obtain asymptotic formulae for the following quantities as $n \to \infty$:
(i) $\displaystyle\sum_{\substack{u \in \mathbb{F}_q[t]^+ \\ \deg(u)=n}} \Lambda(u)$; (ii) $\displaystyle\sum_{\substack{u \in \mathbb{F}_q[t]^+ \\ \deg(u)=n}} \frac{\Lambda(u)}{|u|}$; (iii) $\displaystyle\sum_{\substack{u \in \mathbb{F}_q[t]^+ \\ \deg(u)=n}} \frac{\Lambda^2(u)}{|u|}$.

B5. Let $\sigma(u)$ denote the function $\displaystyle\sum_{\substack{d \in \mathbb{F}_q[t]^+ \\ d|u}} |d|$.

(i) Show that there is a positive constant $C = C(q)$ with the property that, for all $u \in \mathbb{F}_q[t]$ of large degree, one has $\sigma(u) \leqslant C(q)|u| \log_q \log_q |u|$.
(ii) By considering $\sigma(u)$ as a convolution, find a formula for the Dirichlet series $D_\sigma(s)$ in terms of the zeta function $\zeta_{\mathbb{A}}(s)$.
(iii) Obtain a formula for $\displaystyle\sum_{\substack{u \in \mathbb{F}_q[t]^+ \\ \deg(u)=n}} \sigma(u)$.

C6. (i) Show that, for each natural number k, one has $D_{\tau_k}(s) = \zeta_{\mathbb{A}}(s)^k$.

(ii) Prove that for $n \in \mathbb{N}$, one has

$$\sum_{\substack{u \in \mathbb{F}_q[t]^+ \\ \deg(u)=n}} \tau_k(u) = P_k(n) q^n,$$

where $P_k(x)$ is a polynomial of degree $k-1$ in x with leading coefficient $1/(k-1)!$.

(iii) Obtain an asymptotic formula, valid as $n \to \infty$, for

$$\sum_{\substack{u \in \mathbb{F}_q[t]^+ \\ \deg(u)=n}} \frac{\tau_k(u)}{|u|}.$$

C7. (i) Show that when u is a monic polynomial in $\mathbb{A} = \mathbb{F}_q[t]$, one has

$$\frac{|u|}{\varphi(u)} = \sum_{\substack{d \in \mathbb{F}_q[t]^+ \\ d|u}} \frac{\mu^2(d)}{\varphi(d)}.$$

(ii) Prove that as $n \to \infty$, one has

$$\sum_{\substack{u \in \mathbb{F}_q[t]^+ \\ \deg(u)=n}} \frac{|u|}{\varphi(u)} \sim \frac{1-q^{-5}}{(1-q^{-1})(1-q^{-2})} q^n.$$