MA59800ANT ANALYTIC THEORY OF FUNCTION FIELDS. PROBLEMS 3

TO BE HANDED IN BY 6PM WEDNESDAY 2 OCTOBER 2024

Key: A-questions are short questions testing basic skill sets; B-questions integrate essential methods of the course; C-questions are more challenging for enthusiasts, with hints available on request.

Throughout, we take p to be a prime number, put $q = p^h$, and denote $\mathbb{F}_q[t]$ by $\mathbb{A}$.

A1. Recall that an additive character ψ on $\mathbb{F}_q$ is a function $\psi : \mathbb{F}_q \to \mathbb{C}^\times$ having the property that, whenever $a, b \in \mathbb{F}_q$, then $\psi(a+b) = \psi(a)\psi(b)$.
(a) Show that for all $a \in \mathbb{F}_q$, the value of $\psi(a)$ is a p-th root of unity;
(b) Show that $\psi(0) = 1$;
(c) Show that, for all $b \in \{0, 1, 2, \ldots, p-1\}$, one has $\psi(b) = \psi(1)^b$.

A2. Let $\psi : \mathbb{F}_q \to \mathbb{C}^\times$ be an additive character.
(a) Show that, whenever $c \in \mathbb{F}_q$, one has

$$\sum_{u \in \mathbb{F}_q} \psi(u + c) = \psi(c) \sum_{u \in \mathbb{F}_q} \psi(u).$$

(b) Show that for any additive character that is not identically 1, one has

$$\sum_{u \in \mathbb{F}_q} \psi(u) = 0.$$

B3. Suppose that $\{\omega_1, \ldots, \omega_h\}$ is a basis for $\mathbb{F}_q$ over $\mathbb{F}_p$, and let $\psi : \mathbb{F}_q \to \mathbb{C}^\times$ be an additive character.
(a) Show that, for suitable p-th roots of unity $\xi_1, \ldots, \xi_h$, and for all $b_i \in \{0, 1, 2, \ldots, p-1\}$ $(1 \leqslant i \leqslant h)$, one has

$$\psi(b_1\omega_1 + \ldots + b_h\omega_h) = \xi_1^{b_1} \cdots \xi_h^{b_h}.$$

(b) Show that there are q distinct additive characters $\psi : \mathbb{F}_q \to \mathbb{C}^\times$.
(c) Deduce that every additive character $\psi : \mathbb{F}_q \to \mathbb{C}^\times$ has the shape $\psi(u) = e_q(ua)$ for some $a \in \mathbb{F}_q$.

B4. (a) Suppose that $\alpha = \sum_{i \leqslant N} \alpha_i t^i \in \mathbb{F}_q((1/t)) \setminus \{0\}$. Write $\lfloor \alpha \rfloor$ for the polynomial part of α, namely $\sum_{0 \leqslant i \leqslant N} \alpha_i t^i$, and put $\|\alpha\| = |\alpha - \lfloor \alpha \rfloor|$. Use the definition $e(\alpha) = e_q(\operatorname{res} \alpha)$ to deduce that

$$\sum_{\substack{u \in \mathbb{A} \\ |u| < q^n}} e(u\alpha) = \begin{cases} q^n, & \text{when } \|\alpha\| < q^{-n}, \\ 0, & \text{when } \|\alpha\| \geqslant q^{-n}. \end{cases}$$

(b) Prove that

$$\int_{|\alpha|<1} \Bigg| \sum_{\substack{u\in \mathbb{A}\\ |u|<q^n}} e(u\alpha)\Bigg| \,\mathrm{d}\alpha = 1.$$

B5. Let $u \in \mathbb{A}$ have degree exceeding 1.

(a) Show that whenever $c_b \in \mathbb{C}$ $(b \in \mathbb{A})$, then

$$\sum_{\chi\in \mathrm{X}(u)} \Bigg| \sum_{|b|<|u|} c_b\chi(b)\Bigg|^2 = \varphi(u) \sum_{\substack{|b|<|u|\\ (b,u)=1}} |c_b|^2.$$

(b) Show that whenever $c_\chi \in \mathbb{C}$ $(\chi \in \mathrm{X}(u))$, then

$$\sum_{|b|<|u|} \Bigg| \sum_{\chi\in \mathrm{X}(u)} c_\chi\chi(b)\Bigg|^2 = \varphi(u) \sum_{\chi\in \mathrm{X}(u)} |c_\chi|^2.$$

C6. Let $g \in \mathbb{A}$ have degree exceeding 1, and define

$$c_g(u) = \sum_{\substack{|a|<|g|\\ (a,g)=1}} e(au/g).$$

(a) Show that for each fixed $u \in \mathbb{A}$, the function $c_g(u)$ is a multiplicative function of g.

(b) Show that

$$\sum_{d|g} c_d(u) = \begin{cases} |g|, & \text{when } g|u,\\ 0, & \text{otherwise.}\end{cases}$$

(c) Prove that $c_g(u) = \dfrac{\mu(g/(g,u))}{\varphi(g/(g,u))}\varphi(g)$.

C7. Let $\pi \in \mathbb{A}$ be monic and irreducible of degree exceeding 1, and suppose that $k \in \mathbb{N}$ satisfies $k|(|\pi|-1)$. Put

$$f(a) = \sum_{|r|<|\pi|} e(ar^k/\pi).$$

(a) Show that

$$|\pi|^{-1} \sum_{|a|<|\pi|} f(a) = 1 \quad \text{and} \quad |\pi|^{-1} \sum_{|a|<|\pi|} |f(a)|^2 = k(|\pi|-1)+1.$$

(b) Deduce that when $(a,\pi)=1$, one has $|f(a)| \leqslant (k-1)|\pi|^{1/2}$.