NUMBER THEORY: HOMEWORK 1

TO BE HANDED IN THURSDAY 23RD JANUARY 2025 BY 6PM

1. (i) Prove that for every natural number n, one has $(n+3)|(n^3+27)$.
(ii) Suppose that n is a natural number. Prove that

$$(n+1, n^4+n+1) = 1.$$

2. (i) Let a and b be integers. Show that $3|(10a+b)$ if and only if $3|(a+b)$, and hence deduce that an integer n is divisible by 3 if and only if the sum of its base-10 digits is divisible by 3.
(ii) Let a and b be integers. Show that $11|(100a+b)$ if and only if $11|(a+b)$, and hence deduce that an integer n is divisible by 11 if and only if the sum of its base-100 digits is divisible by 11.
(iii) Let a and b be integers. Show that $37|(1000a+b)$ if and only if $37|(a+b)$, and hence deduce that an integer n is divisible by 37 if and only if the sum of its base-1000 digits is divisible by 37.

3. Let the conventional base 10 expansion of the integer n be $n_k n_{k-1} \ldots n_1 n_0$, so that

$$n = 10^k n_k + 10^{k-1} n_{k-1} + \ldots + n_0 \quad \text{with} \quad n_i \in \{0, 1, \ldots, 9\}.$$

Let m be the integer with base 10 expansion $n_k n_{k-1} \ldots n_1$, so that

$$m = 10^{k-1} n_k + 10^{k-2} n_{k-1} + \ldots + n_1.$$

(i) Show that $4n$ (and hence also n) is divisible by 13 if and only if $m + 4n_0$ is divisible by 13, thereby providing a test for divisibility by 13.
(ii) Show that n is divisible by 7 if and only if $m - 2n_0$ is divisible by 7, thereby providing a test for divisibility by 7.

4. Let n be a natural number.
(i) Prove that $(n! - 1, (n+1)! - 1) = 1$.
(ii) Prove that when $n \geqslant 3$, one has $(n! + 2, (n+1)! + 2) = 2$.

5. By considering the binomial coefficient $\binom{n}{k}$, prove that the product of k consecutive integers is always divisible by $k!$.