

NUMBER THEORY: HOMEWORK 1

TO BE HANDED IN THURSDAY 23RD JANUARY 2025 BY 6PM

1. (i) Prove that for every natural number n , one has $(n + 3)|(n^3 + 27)$.

(ii) Suppose that n is a natural number. Prove that

$$(n + 1, n^4 + n + 1) = 1.$$

2. (i) Let a and b be integers. Show that $3|(10a + b)$ if and only if $3|(a + b)$, and hence deduce that an integer n is divisible by 3 if and only if the sum of its base-10 digits is divisible by 3.

(ii) Let a and b be integers. Show that $11|(100a + b)$ if and only if $11|(a + b)$, and hence deduce that an integer n is divisible by 11 if and only if the sum of its base-100 digits is divisible by 11.

(iii) Let a and b be integers. Show that $37|(1000a + b)$ if and only if $37|(a + b)$, and hence deduce that an integer n is divisible by 37 if and only if the sum of its base-1000 digits is divisible by 37.

3. Let the conventional base 10 expansion of the integer n be $n_k n_{k-1} \dots n_1 n_0$, so that

$$n = 10^k n_k + 10^{k-1} n_{k-1} + \dots + n_0 \quad \text{with} \quad n_i \in \{0, 1, \dots, 9\}.$$

Let m be the integer with base 10 expansion $n_k n_{k-1} \dots n_1$, so that

$$m = 10^{k-1} n_k + 10^{k-2} n_{k-1} + \dots + n_1.$$

(i) Show that $4n$ (and hence also n) is divisible by 13 if and only if $m + 4n_0$ is divisible by 13, thereby providing a test for divisibility by 13.

(ii) Show that n is divisible by 7 if and only if $m - 2n_0$ is divisible by 7, thereby providing a test for divisibility by 7.

4. Let n be a natural number.

(i) Prove that $(n! - 1, (n + 1)! - 1) = 1$.

(ii) Prove that when $n \geq 3$, one has $(n! + 2, (n + 1)! + 2) = 2$.

5. By considering the binomial coefficient $\binom{n}{k}$, prove that the product of k consecutive integers is always divisible by $k!$.

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