NUMBER THEORY: HOMEWORK 10

TO BE HANDED IN BY THURSDAY 3RD APRIL 2025 AT 6PM

1. Let $s(n)$ denote the arithmetic function, referred to as the squarefree kernel of n, defined by putting $s(n) = \prod_{p|n} p$.
(a) Show that $s(n)$ is a multiplicative function of n
(b) By applying Möbius inversion, find an arithmetic function $f(n)$ having the property that $s(n) = \sum_{d|n} f(d)$, and write $f(n)$ explicity in terms of familiar arithmetic functions such as $\tau(n)$, $\varphi(n)$, $\mu(n)$, and so on.

2. Show that $\sigma(n) = \sum_{d|n} \varphi(n/d)\tau(d)$.

3. (a) Prove that

$$\sum_{1\leqslant n\leqslant x} \frac{\sigma(n)}{n^2} = \frac{1}{6}\pi^2 \log x + O(1).$$

(b) Obtain an expression, analogous to that given above, for

$$\sum_{1\leqslant n\leqslant x} \frac{\phi(n)}{n^2}.$$

4. (a) Using multiplicativity, prove that

$$\sum_{d=1}^{\infty} \frac{\mu^2(d)}{d^2} = \prod_p (1 + 1/p^2).$$

(b) Hence deduce that

$$\sum_{1\leqslant d\leqslant x} \frac{\mu^2(d)}{d^2} = \frac{15}{\pi^2} + O\Big(\frac{1}{x}\Big).$$

Here, you may assume without proof that

$$\zeta(2) = \prod_p (1 - 1/p^2)^{-1} = \frac{\pi^2}{6} \quad \text{and} \quad \zeta(4) = \prod_p (1 - 1/p^4)^{-1} = \frac{\pi^4}{90}.$$

5. Define $\alpha(n) = \prod_{\substack{1\leqslant a\leqslant n\\(a,n)=1}} a$. Prove that $\alpha(n) = n^{\phi(n)} \prod_{d|n} \left(\frac{d!}{d^d}\right)^{\mu(n/d)}$.