NUMBER THEORY: HOMEWORK 11

TO BE HANDED IN BY THURSDAY 10TH APRIL 2025 BY 6PM

1. (a) Obtain the continued fraction expansions of $\sqrt{5}$ and $\sqrt{6}$.
(b) Obtain the continued fraction expansion of $\sqrt{54}$.

2. Obtain the continued fraction expansions of $\sqrt{69}$ and $\frac{1}{7}(24-\sqrt{15})$.

3. Apply Liouville's Theorem to show that $\theta = \sum_{n=0}^{\infty} 2025^{-n!}$ is transcendental.

4. Let (p_n) be the sequence of prime numbers, so that $p_2 = 2$, $p_3 = 3$, and so on. Define the primorial function $p_n\#$ for each prime number p_n by putting $p_n\# = p_n p_{n-1} \dots 3 \cdot 2$. Prove that the real number $\Theta = \sum_{n=1}^{\infty} 2^{-p_n\#}$ is transcendental.

5. (a) By considering the integral

$$\int_0^1 x^n e^x \, \mathrm{d}x,$$

show that for each natural number n, there are integers A_n and B_n with $A_n e - B_n > 0$ and $\lim_{n\to\infty}(A_n e - B_n) = 0$. Deduce that e is irrational.
(b) By considering the integral

$$\int_{-1}^1 x^n e^x \, \mathrm{d}x,$$

show that e^2 is irrational.