NUMBER THEORY: HOMEWORK 12

TO BE HANDED IN BY THURSDAY 17TH APRIL 2025 BY 6PM

1. Note that the equation $x^2 - 5y^2 = 1$ has a solution $(x, y) = (9, 4)$.
(a) Find a second solution of the equation $x^2 - 5y^2 = 1$ with x and y both positive;
(b) Show that $x^2 - 5y^2 = 1$ has infinitely many integral solutions;
(c) Find a solution of the equation $u^2 - 5v^2 = 5$, and show that there are infinitely many integral solutions of this equation.

2. Recall the continued fraction expansion of $\sqrt{6}$ from Homework 11. Determine the integer solutions to the Pell equation $x^2 - 6y^2 = 1$.

3. Recall the continued fraction expansion of $\sqrt{69}$ from Homework 11. Determine the integer solutions to the Pell equation $x^2 - 69y^2 = 1$.

4. Suppose that d is a positive integer which is not a square. By considering the Pell equation $x^2 - dy^2 = 1$, show that there are infinitely many integers $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(p, q) = 1$ for which one has

$$\left| \sqrt{d} - \frac{p}{q} \right| < \frac{1}{2\sqrt{d}q^2}.$$

5. Let d be a positive integer which is not a perfect square. Prove that, if (x_n, y_n), with $n = 1, 2, ...$ is the sequence of positive solutions of the equation $x^2 - dy^2 = 1$, written according to increasing values of x or y, then x_n and y_n satisfy a recurrence relation $u_{n+2} - au_{n+1} + u_n = 0$, where a is a positive integer.