NUMBER THEORY: HOMEWORK 13

TO BE HANDED IN BY THURSDAY 24TH APRIL 2025 BY 6PM

1. Let k be a positive integer.
(a) Find the fundamental solution of the Pell equation $x^2 - (k^2 + 2)y^2 = 1$, and hence write down a relation that determines all solutions of this equation.
(b) Find two distinct solutions (x_1, y_1) and (x_2, y_2) of the generalised Pell equation $x^2 - (k^2 + 2)y^2 = -2$ in which all x_i and y_i are positive.

2. Let d be a positive integer which is not a perfect square, and consider the negative Pell equation

$$x^2 - dy^2 = -1.$$

(a) Suppose that this equation has distinct non-trivial solutions (x_1, y_1) and (x_2, y_2) with respective associated real numbers $x_1 + y_1\sqrt{d}$ and $x_2 + y_2\sqrt{d}$. Show that the real number

$$x_3 + y_3\sqrt{d} = \frac{x_2 + y_2\sqrt{d}}{x_1 + y_1\sqrt{d}}$$

is associated with a solution (x_3, y_3) of the Pell equation $x^2 - dy^2 = 1$.
(b) Suppose that the fundamental solution of the Pell equation $x^2 - dy^2 = 1$ is (x_0, y_0). Find a relation that defines *all* solutions of the negative Pell equation $x^2 - dy^2 = -1$.

3. Let k be a positive integer.
(a) Find the fundamental solution of the Pell equation $x^2 - (k^2 + 1)y^2 = 1$, and hence write down a relation that determines all solutions of this equation.
(b) Find a solution of the negative Pell equation $x^2 - (k^2 + 1)y^2 = -1$, and hence write down a relation that determines all solutions of this equation.