NUMBER THEORY: HOMEWORK 2

TO BE HANDED IN THURSDAY 30TH JANUARY 2025 BY 6PM

1.(i) Apply the Euclidean algorithm to determine $(3991, 2025)$;

(ii) Find integers x and y such that $3991x + 2025y = (3991, 2025)$;

(iii) Find integers x, y, z such that $15x + 39y + 91z = 1$.

2. Find positive integers a and b satisfying the equations $(a, b) = 111$ and $[a, b] = 999$ simultaneously. Find all solutions.

3.(i) We call an integer *squarefree* if it is not divisible by any integer of the form a^2 with $a > 1$. Show that every positive integer n can be written uniquely in the form $n = ab$ where a is square-free and b is square.

(ii) We call a positive integer n *squarefull* if, whenever p is a prime divisor of n, then p^2 is also a divisor of n. Show that when n is squarefull, there exist positive integers a and b for which $n = a^2b^3$.

4.(i) Prove that there are infinitely many prime numbers of the shape $3k + 2$ for natural numbers k.

(ii) Is it possible that all large primes have the shape $8n \pm 1$? More precisely, does there exist a natural number p_0 with the property that whenever p is a prime number and $p > p_0$, then $p = 8n \pm 1$ for some integer n? Justify your answer.

[Hint: Consider carefully Euclid's proof of the infinitude of primes.]

5* **[Hard].** Let $1 < a_1 < \cdots < a_k < 2n$ be integers *not* dividing each other. Show that $k \leqslant n$. Prove that if $k = n$ and m is the integer satisfying $3^m < 2n < 3^{m+1}$ then $a_1 \geqslant 2^m$.

[Hint: Write each integer a_i in the form $(2b + 1)2^c$. In the second part write $a_1 = (2m_1 + 1)2^r$ and investigate how many numbers a_i must be of the form $(2m_1 + 1)2^c3^d$.]