NUMBER THEORY: HOMEWORK 3

TO BE HANDED IN THURSDAY 6TH FEBRUARY 2025 BY 6PM

1. (i) What are the last three digits in the ordinary decimal expansion of 83^{7601}? What are the last two digits in the expansion of 5^{2025}?

(ii) Show that $17|(2^{5n+4}+7^{2n})$ for all $n \in \mathbb{N}$.

2. (i) Show that $x^3 \equiv 4 \pmod 7$ has no solution. Deduce that $x^3 - 4y^3 = 0$ has no non-zero solution in integers x and y. Deduce that $\sqrt[3]{4}$ is irrational.

(ii) Show that the equation $x^3 - 4y^3 + 14z^3 = 0$ has no non-zero solution in integers x, y and z.

3. (i) Find an integer x such that $3x \equiv 2 \pmod 5$, $2x \equiv 3 \pmod{19}$, and $7x \equiv 5 \pmod 3$. Find an infinite sequence of integers with the same property.

(ii) Find an integer x such that $3x \equiv 2 \pmod 7$, $5x \equiv 3 \pmod{23}$, and $7x \equiv 5 \pmod 9$. Find an infinite sequence of integers with the same property.

(iii) Find all integers x satisfying $2x \equiv 5 \pmod{15}$ and $5x \equiv 7 \pmod{33}$.

4. (i) Recall that if p is prime and $x^2 + 1 \equiv 0 \pmod p$ is soluble, then $p = 2$ or $p \equiv 1 \pmod 4$. By modifying Euclid's proof that there are infinitely many primes, deduce that there are infinitely many primes of the form $4k+1$ ($k \in \mathbb{N}$).

(ii) Show that there are infinitely many primes of the form $8k + 5$ ($k \in \mathbb{N}$).

5. (i) Show that for every natural number n, the least prime **not** dividing n is no larger than $n + 1$. [Hint: consider Euclid's proof that there are infinitely many primes]

(ii) Prove that for each prime number q and each natural number n, one has $q^n \geqslant n + 1$.

(iii) Deduce that if p is the least prime **not** dividing n, then $p - 1$ divides n^n.

(iv) Prove that when n is a natural number, then the least prime **not** dividing n is the smallest prime divisor of $n^{n^n} - 1$.

(v) Let p_k be the k-th smallest prime number, so that $p_1 = 2$, $p_2 = 3$, etc. Prove that p_{k+1} is the smallest prime divisor of $n^{n^n} - 1$, where $n = p_1p_2 \cdots p_k$.