NUMBER THEORY: HOMEWORK 4

TO BE HANDED IN BY THURSDAY 13TH FEBRUARY 2025 BY 6PM

1. (i) Find solutions of $x^2 \equiv -1 \pmod 5$ and $x^2 \equiv -1 \pmod{17}$. Hence, applying the Chinese Remainder Theorem, obtain a solution of $x^2 \equiv -1 \pmod{85}$.

(ii) How many solutions does $x^2 \equiv -1 \pmod{85}$ possess?

2. (i) Let p be a prime number. By applying Fermat's Little Theorem, or otherwise, show that the congruence $x^p - x + 1 \equiv 0 \pmod p$ has no solution.

(ii) How many solutions does the congruence $x^{16} - x + 3 \equiv 0 \pmod{40}$ possess? Explain your answer.

3. By considering the prime factorisation of the integer 561, prove that whenever $(a, 561) = 1$, one has $a^{80} \equiv 1 \pmod{561}$. Hence prove that $a^{560} \equiv 1 \pmod{561}$ whenever $(a, 561) = 1$.

4. Let $f(x) = x^2 + x$ throughout.

(a) Show that for every prime number p and every positive integer k, the congruence $f(x) \equiv 0 \pmod{p^k}$ has precisely 2 solutions.

(b) Let m be a natural number, and let r denote the number of distinct prime numbers dividing m. Show that the congruence $f(x) \equiv 0 \pmod m$ has precisely 2^r solutions.

5. (a) Prove that if p is prime, $(a, p) = 1$ and $(n, p - 1) = 1$, then $x^n \equiv a \pmod p$ has exactly one solution.

(b) Show that when $(n, p - 1) = d$, then $x^n \equiv 1 \pmod p$ has precisely d solutions.