NUMBER THEORY: HOMEWORK 8

TO BE HANDED IN BY THURSDAY 13TH MARCH 2025 BY 6PM

1. Calculate the symbols $\left(\frac{264}{173}\right)$, $\left(\frac{2019}{4987}\right)$, $\left(\frac{187}{389}\right)$.

2. (a) Determine the odd prime numbers p for which 5 is a quadratic residue modulo p.

(b) By considering the polynomial $x^2 - 5$, and applying a variant of Euclid's proof, show that there are infinitely many primes of the shape $5k + 4$.

3. Let p be an odd prime number. Determine the primes p for which -7 is a quadratic residue modulo p.

4. (a) Show that $\left(\frac{3}{p}\right) = -1$ whenever $p \equiv 5 \pmod{12}$.

(b) Suppose that $p = 2^{2^n} + 1$ is a prime number. Show that 3 is a primitive root modulo p.

5. Use the following strategy to prove that there are no integers x, y satisfying the equation $y^2 = x^3 + 45$.

(a) Show that if (x, y) were to satisfy this equation, then $x \equiv 7 \pmod 8$ or $x \equiv 3 \pmod 8$.

(b) If $x \equiv 7 \pmod 8$, rewrite the equation as

$$y^2 - 2 \cdot 3^2 = (x + 3)(x^2 - 3x + 9).$$

Prove that $x^2 - 3x + 9$ must be divisible by a prime $p \equiv \pm 3 \pmod 8$, and derive a contradiction.

(c) Deal with the second case in which $x \equiv 3 \pmod 8$ by writing $y^2 = x^3 + 45$ as

$$y^2 - 2 \cdot 6^2 = (x - 3)(x^2 + 3x + 9),$$

and proceeding similarly.