NUMBER THEORY: HOMEWORK 9

TO BE HANDED IN BY TUESDAY 25TH MARCH 2025

1. (a) Prove that for all natural numbers n, one has $\sum_{d|n} \mu^2(d) = 2^{\omega(n)}$.

(b) Prove that for all natural numbers n, one has $\sum_{d|n} \mu(d)\tau(d) = (-1)^{\omega(n)}$.

2. (a) Show that

$$\sum_{a=1}^{n} a^3 = \tfrac{1}{4}n^2(n+1)^2 = \left(\sum_{a=1}^{n} a \right)^2 .$$

(b) Prove that for all natural numbers n, one has $\sum_{d|n} \tau(d)^3 = \left(\sum_{d|n} \tau(d) \right)^2$.

3. Let f be an arithmetic function.

(a) Prove that when a and n are positive integers, then

$$\sum_{d|(a,n)} \mu(d) = \begin{cases} 1, & \text{when } (a,n) = 1, \\ 0, & \text{when } (a,n) > 1. \end{cases}$$

(b) Deduce that

$$\sum_{\substack{1 \leqslant a \leqslant n \\ (a,n)=1}} f(a) = \sum_{d|n} \mu(d) \sum_{\substack{1 \leqslant a \leqslant n \\ d|a}} f(a).$$

(c) Prove that when $n > 1$, one has

$$\sum_{\substack{1 \leqslant a \leqslant n \\ (a,n)=1}} 1 = \varphi(n) \quad \text{and} \quad \sum_{\substack{1 \leqslant a \leqslant n \\ (a,n)=1}} a = \tfrac{1}{2}n\varphi(n).$$

©Trevor D. Wooley, Purdue University 2025. This material is copyright of Trevor D. Wooley at Purdue University unless explicitly stated otherwise. It is provided exclusively for educational purposes at Purdue University, and is to be downloaded or copied for your private study only.