SOLUTIONS TO HOMEWORK 1

1. (i) When $n \in \mathbb{N}$, one has $n^3 + 27 = (n+3)(n^2 - 3n + 9)$, and so $n + 3$ divides $n^3 + 27$, as required.
(ii) When $n \geqslant 1$, one has

$$(n+1, n^4 + n + 1) = (n+1, n^4 + n + 1 - (n+1)(n^3 - n^2 + n)),$$

and thus $(n + 1, n^4 + n + 1) = (n + 1, 1) = 1$.

2. (i) One has $3|(10a+b)$ if and only if $3|(10a+b-9a)$, or equivalently $3|(a+b)$. Write $n = 10^k n_k + 10^{k-1} n_{k-1} + \ldots + n_0$ in the ordinary base-10 expansion. Using the above conclusion, one finds that $3|n$ if and only if

$$3|(10^{k-1} n_k + \ldots + 10n_2 + n_1 + n_0),$$

or equivalently $3|(10^{k-2} n_k + \ldots + 10n_3 + n_2 + (n_1 + n_0))$, and so on. Thus, by induction, one sees that $3|n$ if and only if $3|(n_k + n_{k-1} + \ldots + n_0)$, as required.
(ii) One has $11|(100a + b)$ if and only if $11|(100a + b - 9(11a))$, or equivalently $11|(a + b)$. Write $n = 100^k n_k + 100^{k-1} n_{k-1} + \ldots + n_0$ in the ordinary base-100 expansion. Using the above conclusion, one finds that $11|n$ if and only if $11|(100^{k-1} n_k + \ldots + 100n_2 + n_1 + n_0)$, or equivalently

$$11|(100^{k-2} n_k + \ldots + 100n_3 + n_2 + (n_1 + n_0)),$$

and so on. Thus, one sees that $11|n$ if and only if $11|(n_k + n_{k-1} + \ldots + n_0)$, as required.
(iii) One has $37|(1000a+b)$ if and only if $37|(1000a+b-27(37a))$, or equivalently $37|(a + b)$. Write $n = 1000^k n_k + 1000^{k-1} n_{k-1} + \ldots + n_0$ in the ordinary base-1000 expansion. Using the above conclusion, one finds that $37|n$ if and only if $37|(1000^{k-1} n_k + \ldots + 1000n_2 + n_1 + n_0)$, or equivalently

$$37|(1000^{k-2} n_k + \ldots + 1000n_3 + n_2 + (n_1 + n_0)),$$

and so on. Thus, one sees that $37|n$ if and only if $37|(n_k + n_{k-1} + \ldots + n_0)$, as required.

3. (i) Since 4 and 13 are coprime, one finds that $n = 10m + n_0$ is divisible by 13 if and only if $4n = 40m + 4n_0$ is divisible by 13. But the latter holds if and only if $40m + 4n_0 - 39m = m + 4n_0$ is divisible by 13. Thus $13|n$ if and only if $m + 4n_0$ is divisible by 13, as required.
(ii) Since 2 and 7 are coprime, one finds that $n = 10m + n_0$ is divisible by 7 if and only if $-2n = -20m - 2n_0$ is divisible by 7. But the latter holds if and only if $-20m - 2n_0 + 3(7m) = m - 2n_0$ is divisible by 7. Thus $7|n$ if and only if $m - 2n_0$ is divisible by 7, as required.

4. (i) One has $(n! - 1, (n+1)! - 1) = (n! - 1, ((n+1)! - 1) - (n+1)(n! - 1)) = (n! - 1, n)$. But $(n! - 1, n) = (n! - 1 - n \cdot (n-1)!, n) = (-1, n) = 1$, and so $(n! - 1, (n+1)! - 1) = 1$, as required.

(ii) When $n \geqslant 3$, one has

$$(n!+2,(n+1)!+2) = (n!+2,((n+1)!+2)-(n+1)(n!+2)) = (n!+2,-2n).$$

But since when $n \geqslant 3$, one has $2|(n-1)!$, it follows that

$$(n!+2,-2n) = (n!+2-2n\cdot\tfrac{1}{2}(n-1)!,-2n) = (2,-2n) = 2(1,n) = 2,$$

and so $(n!+2,(n+1)!+2)=2$, as required.

5. If the k consecutive integers in question contain 0, then this conclusion is trivial. Also, when all k integers are negative, then their product is equal to $(-1)^k$ multiplied by the product of k consecutive positive integers, and thus there is no loss of generality in restricting to the case of k consecutive positive integers. Whenever $k,n\in\mathbb{N}$ satisfy $k\leqslant n$, one has

$$\frac{n(n-1)\cdots(n-k+1)}{k!} = \binom{n}{k} \in \mathbb{N},$$

and hence $k!$ divides $n(n-1)\cdots(n-k+1)$. Then the product of any k positive integers is divisible by $k!$, and this completes the proof.