SOLUTIONS TO HOMEWORK 11

1. (a) One has

$$[\sqrt{5}] = 2, \quad 1/(\sqrt{5}-2) = \sqrt{5}+2,$$
$$[\sqrt{5}+2] = 4, \quad 1/((\sqrt{5}+2)-4) = 1/(\sqrt{5}-2) = \sqrt{5}+2,$$

and we obtain repetition. Thus $\sqrt{5} = [2; \overline{4}]$.

Also, one has

$$[\sqrt{6}] = 2, \quad 1/(\sqrt{6}-2) = (\sqrt{6}+2)/2,$$
$$[(\sqrt{6}+2)/2] = 2, \quad 1/((\sqrt{6}+2)/2-2) = 2/(\sqrt{6}-2) = \sqrt{6}+2,$$
$$[\sqrt{6}+2] = 4, \quad 1/((\sqrt{6}+2)-4) = 1/(\sqrt{6}-2) = (\sqrt{6}+2)/2,$$

and we obtain repetition. Thus $\sqrt{6} = [2; \overline{2,4}]$.

(b) One has

$$[\sqrt{54}] = 7, \quad 1/(\sqrt{54}-7) = (\sqrt{54}+7)/5,$$
$$[(\sqrt{54}+7)/5] = 2, \quad 1/((\sqrt{54}+7)/5-2) = 5/(\sqrt{54}-3) = (\sqrt{54}+3)/9,$$
$$[(\sqrt{54}+3)/9] = 1, \quad 1/((\sqrt{54}+3)/9-1) = 9/(\sqrt{54}-6) = (\sqrt{54}+6)/2,$$
$$[(\sqrt{54}+6)/2] = 6, \quad 1/((\sqrt{54}+6)/2-6) = 2/(\sqrt{54}-6) = (\sqrt{54}+6)/9,$$
$$[(\sqrt{54}+6)/9] = 1, \quad 1/((\sqrt{54}+6)/9-1) = 9/(\sqrt{54}-3) = (\sqrt{54}+3)/5,$$
$$[(\sqrt{54}+3)/5] = 2, \quad 1/((\sqrt{54}+3)/5-2) = 5/(\sqrt{54}-7) = \sqrt{54}+7,$$
$$[\sqrt{54}+7] = 14, \quad 1/((\sqrt{54}+7)-14) = 1/(\sqrt{54}-7) = (\sqrt{54}+7)/5,$$

and we obtain repetition. Thus $\sqrt{54} = [7; \overline{2,1,6,1,2,14}]$.

2. One has

$$[\sqrt{69}] = 8, \quad 1/(\sqrt{69}-8) = (\sqrt{69}+8)/5,$$
$$[(\sqrt{69}+8)/5] = 3, \quad 1/((\sqrt{69}+8)/5-3) = 5/(\sqrt{69}-7) = (\sqrt{69}+7)/4,$$
$$[(\sqrt{69}+7)/4] = 3, \quad 1/((\sqrt{69}+7)/4-3) = 4/(\sqrt{69}-5) = (\sqrt{69}+5)/11,$$
$$[(\sqrt{69}+5)/11] = 1, \quad 1/((\sqrt{69}+5)/11-1) = 11/(\sqrt{69}-6) = (\sqrt{69}+6)/3,$$
$$[(\sqrt{69}+6)/3] = 4, \quad 1/((\sqrt{69}+6)/3-4) = 3/(\sqrt{69}-6) = (\sqrt{69}+6)/11,$$
$$[(\sqrt{69}+6)/11] = 1, \quad 1/((\sqrt{69}+6)/11-1) = 11/(\sqrt{69}-5) = (\sqrt{69}+5)/4,$$
$$[(\sqrt{69}+5)/4] = 3, \quad 1/((\sqrt{69}+5)/4-3) = 4/(\sqrt{69}-7) = (\sqrt{69}+7)/5,$$
$$[(\sqrt{69}+7)/5] = 3, \quad 1/((\sqrt{69}+7)/5-3) = 5/(\sqrt{69}-8) = \sqrt{69}+8,$$
$$[\sqrt{69}+8] = 16, \quad 1/(\sqrt{69}-8) = (\sqrt{69}+8)/5,$$

and we obtain repetition. Thus $\sqrt{69} = [8; \overline{3,3,1,4,1,3,3,16}]$.

Also, one has

$$[(24-\sqrt{15})/7] = 2, \quad 1/((24-\sqrt{15})/7-2) = 7/(10-\sqrt{15}) = 7(10+\sqrt{15})/85,$$
$$[7(10+\sqrt{15})/85] = 1, \quad 1/(7(10+\sqrt{15})/85-1) = 85/(-15+7\sqrt{15}) = (15+7\sqrt{15})/6,$$

$[(15+7\sqrt{15})/6]=7, \quad 1/((15+7\sqrt{15})/6-7)=6/(-27+7\sqrt{15})=27+7\sqrt{15},$
$[27+7\sqrt{15}]=54, \quad 1/((27+7\sqrt{15})-54)=1/(-27+7\sqrt{15})=(27+7\sqrt{15})/6,$
$[(27+7\sqrt{15})/6]=9, \quad 1/((27+7\sqrt{15})/6-9)=6/(-27+7\sqrt{15})=27+7\sqrt{15},$
and we obtain repetition. Thus $(24-\sqrt{15})/7=[2;1,7,\overline{54,9}]$.

3. Write $\theta=\sum_0^\infty 2025^{-n!}$. For each natural number j, write $q_j=2025^{j!}$ and

$$a_j=2025^{j!}\sum_{n=0}^{j}2025^{-n!}.$$

Then both a_j and q_j are natural numbers with $(a_j,q_j)=1$, and

$$|\theta-a_j/q_j|=\sum_{n=j+1}^{\infty}2025^{-n!}<2025^{1-(j+1)!}<q_j^{-j}.$$

If θ were algebraic, then it would be algebraic of some degree $d\geqslant 1$. By Liouville's theorem, for some positive number c, one would have $|\theta-a/q|>c/q^d$ for every pair of natural numbers a and q with $(a,q)=1$ and q large enough. But the above upper bound contradicts this lower bound as soon as $j>d$ and j is large enough in terms of c. Hence θ is transcendental.

4. Write $\Theta=\sum_1^\infty 2^{-p_n\#}$. For each natural number j, write $q_j=2^{p_j\#}$ and

$$a_j=2^{p_j\#}\sum_{n=1}^{j}2^{-p_n\#}.$$

Then both a_j and q_j are natural numbers with $(a_j,q_j)=1$, and

$$|\Theta-a_j/q_j|=\sum_{n=j+1}^{\infty}2^{-p_n\#}<2^{1-p_{j+1}\#}<q_j^{-j}.$$

Notice here that we use the trivial lower bound $p_{j+1}\geqslant j+1$ to derive the last of these inequalities. If Θ were algebraic, then it would be algebraic of some degree $d\geqslant 1$. By Liouville's theorem, for some positive number c, one would have $|\Theta-a/q|>c/q^d$ for every pair of natural numbers a and q with $(a,q)=1$ and q large enough. But the above upper bound contradicts this lower bound as soon as $j>d$ and j is large enough in terms of c. Hence Θ is transcendental.

5. (a) Write

$$I_n=\int_0^1 x^n e^x\,\mathrm{d}x.$$

Then, by integrating by parts, one finds that when $n\geqslant 1$, one has

$$I_n=[x^n e^x]_0^1-n\int_0^1 x^{n-1}e^x\,\mathrm{d}x=e-nI_{n-1}.$$

Meanwhile, one has $I_0=e-1$. Then it follows by induction that for each natural number n, there are integers A_n and B_n for which $I_n=A_ne-B_n$. On the other hand, the positivity of the integrand ensures that $A_ne-B_n=I_n>0$.

Finally, when $0 < x < 1$, one has $0 < x^n e^x < ex^n$, so that $x^n e^x \to 0$ as $n \to \infty$. It follows that

$$\lim_{n\to\infty} \int_0^1 x^n e^x \,\mathrm{d}x = 0,$$

and hence $\lim_{n\to\infty}(A_n e - B_n) = \lim_{n\to\infty} I_n = 0$. However, were e to be rational, then one would have natural numbers b and k with $(b,k) = 1$ such that $e = b/k$, and then since $A_n e - B_n > 0$, we see that $A_n e - B_n \geqslant 1/k$. This shows that $\liminf_{n\to\infty}(A_n e - B_n) \geqslant 1/k$, in contradiction with the conclusion $\lim_{n\to\infty}(A_n e - B_n) = 0$. Hence e must be irrational.

(b) Write

$$J_n = \int_{-1}^1 x^n e^x \,\mathrm{d}x.$$

Then, by integrating by parts, one finds that when $n \geqslant 1$, one has

$$J_n = [x^n e^x]_{-1}^1 - n \int_{-1}^1 x^{n-1} e^x \,\mathrm{d}x = e - (-1)^n e^{-1} - nJ_{n-1}.$$

Meanwhile, one has $J_0 = e - e^{-1}$. Then it follows by induction that for each natural number n, there are integers C_n and D_n for which $J_n = C_n e - D_n e^{-1}$. On the other hand, the positivity of the integrand ensures that $C_n e - D_n e^{-1} = J_n > 0$. Finally, when $-1 < x < 1$, one has $0 < |x^n e^x| < e|x|^n$, so that $x^n e^x \to 0$ as $n \to \infty$. It follows that

$$\lim_{n\to\infty} \int_{-1}^1 x^n e^x \,\mathrm{d}x = 0,$$

and hence $\lim_{n\to\infty}(C_n e^2 - D_n) = e \lim_{n\to\infty} J_n = 0$. However, were e^2 to be rational, then one would have natural numbers c and l with $(c,l) = 1$ such that $e^2 = c/l$, and then since $C_n e^2 - D_n > 0$, we see that $C_n e^2 - D_n \geqslant 1/l$. This shows that $\liminf_{n\to\infty}(C_n e^2 - D_n) \geqslant 1/l$, in contradiction with the conclusion $\lim_{n\to\infty}(C_n e^2 - D_n) = 0$. Hence e^2 must be irrational.