SOLUTIONS TO HOMEWORK 12

1. (a) Since the equation $x^2 - 5y^2 = 1$ has the solution $(x, y) = (9, 4)$, we know that $(x, y) = (9^2 + 5 \cdot 4^2, 2 \cdot 9 \cdot 4) = (161, 72)$ also solves this equation.
(b) Suppose that $x^2 - 5y^2 = 1$ has just finitely many solutions, and let (x, y) be the solution with x largest. Then $(x^2 + 5y^2, 2xy)$ is a solution with $x^2 + 5y^2 > x$, giving a contradiction. So the equation has infinitely many integral solutions.
(c) One can check that $5^2 - 5 \cdot 2^2 = 25 - 20 = 5$, and so $(u, v) = (5, 2)$ solves $u^2 - 5v^2 = 5$. Suppose that (x, y) is a solution of $x^2 - 5y^2 = 1$. Consider the real number $(5 + 2\sqrt{5})(x + y\sqrt{5}) = (5x + 10y) + (2x + 5y)\sqrt{5}$. Motivated by multiplication by the conjugate, one finds that

$$(5x + 10y)^2 - 5(2x + 5y)^2 = 5(x^2 - 5y^2) = 5.$$

Since from part (b) there are infinitely many solutions (x, y) of $x^2 - 5y^2 = 1$, then there are infinitely many solutions $(u, v) = (5x + 10y, 2x + 5y)$ of the equation $u^2 - 5v^2 = 5$.

2. Recall that $\sqrt{6} = [2; \overline{2, 4}]$. We compute the convergents p_n/q_n to the continued fraction expansion of $\sqrt{6}$, using the recurrence relations from class:

$$\frac{p_0}{q_0} = \frac{2}{1} \quad \text{and} \quad p_0^2 - 6q_0^2 = 2^2 - 6 \cdot 1^2 = -2,$$
$$\frac{p_1}{q_1} = \frac{2 \cdot 2 + 1}{2} \quad \text{and} \quad p_1^2 - 6q_1^2 = 5^2 - 6 \cdot 2^2 = 1,$$

so the fundamental solution of $x^2 - 6y^2 = 1$ is $(x, y) = (5, 2)$. Thus, every solution (x, y) of $x^2 - 6y^2 = 1$ is given by $x + y\sqrt{6} = \pm(5 + 2\sqrt{6})^n$ $(n \in \mathbb{Z})$.

3. From question 2 of Homework 11, we have $\sqrt{69} = [8; \overline{3, 3, 1, 4, 1, 3, 3, 16}]$. This continued fraction is periodic with period 8. Writing p_n/q_n for the convergents to the continued fraction expansion of $\sqrt{69}$, we find that the fundamental solution is given by (p_7, q_7) (notice that this index 7 is odd). Using recurrence relations from class, we have

$$\frac{p_0}{q_0} = \frac{8}{1},$$
$$\frac{p_1}{q_1} = \frac{3 \cdot 8 + 1}{3} = \frac{25}{3},$$
$$\frac{p_2}{q_2} = \frac{3 \cdot 25 + 8}{3 \cdot 3 + 1} = \frac{83}{10},$$
$$\frac{p_3}{q_3} = \frac{1 \cdot 83 + 25}{1 \cdot 10 + 3} = \frac{108}{13},$$
$$\frac{p_4}{q_4} = \frac{4 \cdot 108 + 83}{4 \cdot 13 + 10} = \frac{515}{62},$$

$$\frac{p_5}{q_5} = \frac{1 \cdot 515 + 108}{1 \cdot 62 + 13} = \frac{623}{75},$$
$$\frac{p_6}{q_6} = \frac{3 \cdot 623 + 515}{3 \cdot 75 + 62} = \frac{2384}{287},$$
$$\frac{p_7}{q_7} = \frac{3 \cdot 2384 + 623}{3 \cdot 287 + 75} = \frac{7775}{936},$$

so the fundamental solution of $x^2 - 69y^2 = 1$ is $(x, y) = (7775, 936)$. Indeed, one has

$$7775^2 = 60450625$$
$$69 \cdot 936^2 = 60450624.$$

Thus, the solutions (x, y) of $x^2 - 69y^2 = 1$ are given by

$$x + y\sqrt{69} = \pm(7775 + 936\sqrt{69})^n \quad (n \in \mathbb{Z}).$$

It is now easy to compute examples of further solutions. For example, since

$$(7775 + 936\sqrt{69})^2 = 7775^2 + 69 \cdot 936^2 + 2 \cdot 7775 \cdot 936\sqrt{69},$$

we find that $(x, y) = (120901249, 14554800)$ is the second smallest solution of $x^2 - 69y^2 = 1$.

4. We know that there are infinitely many integral solutions (x, y) to the Pell equation $x^2 - dy^2 = 1$ with $x > 1$ and $y > 1$. In particular, we have $(x, y) = 1$ and $x = \sqrt{dy^2 + 1} > y\sqrt{d}$. Thus, since $(y\sqrt{d} - x)(y\sqrt{d} + x) = -1$, we see that

$$|y\sqrt{d} - x| = \frac{1}{|y\sqrt{d} + x|} < \frac{1}{2y\sqrt{d}},$$

whence $|\sqrt{d} - x/y| < 1/(2\sqrt{d}y^2)$. Since there are infinitely many such pairs (x, y), the desired conclusion follows.

5. Let (p, q) denote the solution of $x^2 - dy^2 = 1$ with $p, q \in \mathbb{N}$ and with p, q smallest. Then the set of all solutions of $x^2 - dy^2 = 1$ is given by $\pm(A_m, B_m)$ with $m \in \mathbb{Z}$, where A_m, B_m are the integers determined from the relation $A_m + B_m\sqrt{d} = (p + q\sqrt{d})^m$. In particular, these solutions satisfy $A_{m+1} > A_m$ for $m \in \mathbb{N}$, and similarly $B_{m+1} > B_m$. Also, one has

$$A_{m+1} + B_{m+1}\sqrt{d} = (p + q\sqrt{d})(A_m + B_m\sqrt{d}) = (pA_m + dqB_m) + \sqrt{d}(qA_m + pB_m),$$

and

$$\begin{aligned} A_{m+2} + B_{m+2}\sqrt{d} &= (p + q\sqrt{d})^2(A_m + B_m\sqrt{d}) \\ &= ((p^2 + q^2 d)A_m + 2dpqB_m) + \sqrt{d}(2pqA_m + (p^2 + dq^2)B_m). \end{aligned}$$

Consequently,

$$\begin{aligned} A_{m+2} - 2pA_{m+1} &= ((p^2 + dq^2)A_m + 2pqdB_m) - 2p(pA_m + dqB_m) \\ &= (dq^2 - p^2)A_m = -A_m, \end{aligned}$$

and

$$\begin{aligned} B_{m+2} - 2pB_{m+1} &= ((p^2 + dq^2)B_m + 2pqA_m) - 2p(qA_m + pB_m) \\ &= (dq^2 - p^2)B_m = -B_m. \end{aligned}$$

So the sequence of positive solutions (x_n, y_n) of $x^2 - dy^2 = 1$, written according to increasing values of x or y, satisfies $u_{n+2} - 2pu_{n+1} + u_n = 0$ ($u = x$ or y), where p is the smallest positive integer such that $p^2 - dq^2 = 1$ is soluble with $q \in \mathbb{N}$.