SOLUTIONS TO HOMEWORK 9

1. (a) The function $\mu(n)$ is multiplicative, and hence $\mu^2(n)$ is also multiplicative. Then it suffices to examine prime powers, where we find that for each prime p and non-negative integer h, one has

$$\sum_{d|p^h} \mu^2(d) = \sum_{l=0}^{h} \mu^2(p^l) = \begin{cases} 1, & \text{when } h = 0, \\ 1 + \mu(p)^2 = 2, & \text{when } h \geqslant 1. \end{cases}$$

Thus, by applying multiplicativity, we see that when $n = \prod_{p^h \| n} p^h$, one has $\sum_{d|n} \mu^2(d) = \prod_{p|n} 2 = 2^{\omega(n)}$, as required.
(b) Since $\tau(n)$ is also multiplicative, we may proceed in like manner. Here we note that $\tau(p^l) = l + 1$, and hence

$$\sum_{d|p^h} \mu(d)\tau(d) = \sum_{l=0}^{h} \mu(p^l)\tau(p^l) = \begin{cases} 1, & \text{when } h = 0, \\ 1 - 2 = -1, & \text{when } h \geqslant 1. \end{cases}$$

Thus, by applying multiplicativity, we see that when $n = \prod_{p^h \| n} p^h$, one has $\sum_{d|n} \mu(d)\tau(d) = \prod_{p|n}(-1) = (-1)^{\omega(n)}$, as required.
2. (a) The sum of the first n positive integers is $n(n + 1)/2$, so

$$\left(\sum_{a=1}^{n} a\right)^2 = (n(n + 1)/2)^2 = \tfrac{1}{4}n^2(n + 1)^2.$$

Meanwhile, whenever

$$\sum_{a=1}^{n} a^3 = \tfrac{1}{4}n^2(n + 1)^2,$$

then one has

$$\sum_{a=1}^{n+1} a^3 = (n+1)^3 + \tfrac{1}{4}n^2(n+1)^2 = \tfrac{1}{4}(n+1)^2(4(n+1)+n^2) = \tfrac{1}{4}(n+1)^2(n+2)^2.$$

Since $\sum_{a=1}^{1} a^3 = 1 = \frac{1}{4}1^2(1 + 1)^2$, we conclude by induction that

$$\sum_{a=1}^{n} a^3 = \tfrac{1}{4}n^2(n + 1)^2 = \left(\sum_{a=1}^{n} a\right)^2.$$

(b) For each prime power p^h, we have

$$\sum_{a=0}^{h} \tau(p^a) = \sum_{a=0}^{h}(a + 1) = \tfrac{1}{2}(h + 1)(h + 2),$$

and

$$\sum_{a=0}^{h} \tau(p^a)^3 = \sum_{a=0}^{h} (a+1)^3 = \tfrac{1}{4}(h+1)^2(h+2)^2.$$

Thus, whenever n is a prime power, one has

$$\sum_{d|n} \tau(d)^3 = \left(\sum_{d|n} \tau(d) \right)^2,$$

and the desired conclusion follows by multiplicativity.

3. Let f be an arithmetic function.

(a) When a and n are positive integers, one has

$$\sum_{d|(a,n)} \mu(d) = \nu((a,n)) = \begin{cases} 1, & \text{when } (a,n) = 1, \\ 0, & \text{when } (a,n) > 1. \end{cases}$$

(b) Thus

$$\sum_{\substack{1 \leqslant a \leqslant n \\ (a,n)=1}} f(a) = \sum_{1 \leqslant a \leqslant n} \sum_{d|(a,n)} \mu(d) f(a) = \sum_{d|n} \mu(d) \sum_{\substack{1 \leqslant a \leqslant n \\ d|a}} f(a).$$

(c) We now assume that $n > 1$. First taking $f(a) = 1$, we find that

$$\sum_{\substack{1 \leqslant a \leqslant n \\ (a,n)=1}} 1 = \sum_{d|n} \mu(d) \sum_{\substack{1 \leqslant a \leqslant n \\ d|a}} 1 = \sum_{d|n} \mu(d) n/d = \varphi(n).$$

Next, taking $f(a) = a$, we obtain

$$\begin{aligned} \sum_{\substack{1 \leqslant a \leqslant n \\ (a,n)=1}} a = \sum_{d|n} \mu(d) \sum_{\substack{1 \leqslant a \leqslant n \\ d|a}} a &= \sum_{d|n} \mu(d) d \cdot \tfrac{1}{2}(n/d)(n/d+1) \\ &= \tfrac{1}{2} n \sum_{d|n} \mu(d) n/d + \tfrac{1}{2} n \sum_{d|n} \mu(d) = \tfrac{1}{2} n \varphi(n). \end{aligned}$$

As a quick alternative, one can also use the change of summation $a \mapsto n - a$ to see that

$$2 \sum_{\substack{1 \leqslant a \leqslant n \\ (a,n)=1}} a = \sum_{\substack{1 \leqslant a \leqslant n \\ (a,n)=1}} a + \sum_{\substack{1 \leqslant a \leqslant n \\ (a,n)=1}} (n-a) = n \sum_{\substack{1 \leqslant a \leqslant n \\ (a,n)=1}} 1 = n\varphi(n),$$

from which the desired conclusion is immediate.