FURTHER IMPROVEMENTS IN WARING'S PROBLEM.

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Contents

1. Introduction

In this paper we are concerned with sums of k^{th} powers for k in the range $5 \leq k \leq 15$. As usual, we let $G(k)$ denote the smallest number s such that every sufficiently large natural number is the sum of, at most, s k^{th} powers of natural numbers. The last few years have seen remarkable progress in the stubborn problem of reducing the upper bound for $G(k)$; in Table 1.1 we display the upper bounds for $G(k)$ which have been obtained recently in the range considered here.

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Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX

k	5	6	7	8	9	10	11	12	13	14	15
Vaughan [**6**, **7**]	21	31	45	62	82						
Vaughan [**8**, **9**]	19	29	41	57	75	93	109	125	141	156	171
Brüdern [**1**]	18										
Vaughan and Wooley [**10**]	18	28				92	108	124	139	153	168
Wooley [**13**]		27	36	47	55	63	70	79	87	95	103

Table 1.1.

By exploiting the flexibility of the new iterative methods in Waring's problem, we now achieve the following bounds.

Theorem 1.1. *$G(5) \le 17$, $G(6) \le 25$, $G(7) \le 33$, $G(8) \le 43$, $G(9) \le 51$.*

The calculations involved in the proofs are decidedly heavy, especially in the exceptionally awkward case $k = 6$, and in general grow steadily with k. However, for larger k there is an increasingly common pattern. Thus, whilst we have not exhaustively analysed for such k all possible variants of our methods, we have performed sufficient calculations to establish, in combination with results in [**12**] and [**16**], the upper bounds $G(10) \le 59$, $G(11) \le 67$, $G(12) \le 76$, $G(13) \le 84$, $G(14) \le 92$, $G(15) \le 100$.

There are many applications of the methods we develop, these depending on the underlying mean value theorems. For example, we are able to improve results on the distribution of fractional parts of sequences αn^k, and on the solubility of systems of simultaneous additive equations. We intend pursuing some of these applications in a future memoir. Furthermore, we have found some rather technical refinements which permit the above bounds for $G(k)$ to be improved when $k = 6$ and $k = 8$. Thus, in the sequel papers [**11**] and [**12**], we describe some delicate innovations which permit the mean values of this paper to be slightly better exploited, thereby establishing the bounds $G(6) \le 24$ and $G(8) \le 42$.

As is usual in much of the modern work on Waring's problem, the method is dependent on upper bounds for the number of solutions of auxiliary equations of the type

$$x_1^k + \cdots + x_s^k = y_1^k + \cdots + y_s^k, \tag{1.1}$$

with $x_i, y_i \in \mathcal{A}(P, R)$, where throughout we write

$$\mathcal{A}(P, R) = \{1 \le n \le P : p \text{ prime}, \ p|n \text{ implies } p \le R\}.$$

In Wooley [**13**] an improvement over the strategy of Vaughan [**8**, **9**] is established which, through the use of more efficient differences, enables one to obtain better estimates than have been obtained hitherto for the number of solutions of (1.1) when $k \ge 6$. In that memoir, no attempt was made to exploit the finer properties of the polynomials arising from the efficient differencing procedure. Furthermore, the underlying themes of this improved strategy permit a more flexible approach than was employed therein. In this paper we take advantage of this greater flexibility in a number of ways. This requires the exponential sums arising from the efficient differencing procedure to be examined in some detail with regard to their second and fourth moments, and their supremum on appropriate choices of minor arcs. This we do in §§3 and 4 respectively. In this way we are able to obtain satisfactory bounds for the number of solutions of (1.1) for appropriate ranges of k and s.

In order to set the overall pattern we first of all treat fifth powers. In §§5 and 6 we apply the results of §§3 and 4 respectively. In the final iteration of the method, we are presented with the recurring problem that, in our estimate for the number of solutions of equation (1.1), the dominant contribution arises from the "major arcs". We overcome this obstacle in §7 by modifying the arguments of Vaughan and Wooley [**10**]. Having illustrated the framework of our method with fifth powers, we apply the results of §3 to higher values of k in §§8, 9, 10 and 11. It then remains to complete our arguments by applying the results of §4. Thus we consider sixth powers in §12. In §13 we consider some rather general arguments of use in the Hardy-Littlewood dissections used for larger k. Finally the values $k = 7, 8, 9$ are treated in §§14, 15, and 16 respectively.

Before proceeding to the details, in §2 below we describe the strategies which underly our new analysis, and also introduce some notation.

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2. Preliminary lemmata

The methods we adopt lead to more complex iterative processes than have been used in Waring's problem hitherto. We take this opportunity to explain the underlying themes in a little detail for k an arbitrary integer exceeding 2. First we shall establish some notation, which we use in this section and in those following.

Throughout, s will denote a positive integer, and ε and η will denote sufficiently small positive numbers. We take P to be a large positive real number depending at most on k, s, ε and η. We use $\ll$ and $\gg$ to denote Vinogradov's well-known notation, implicit constants depending at most on k, s, ε and η. We make frequent use of vector notation for brevity. For example, $(c_1, \dots, c_t)$ is abbreviated to $\mathbf{c}$. Also, we shall write $e(\alpha)$ for $e^{2\pi i\alpha}$, and $[x]$ for the greatest integer not exceeding x. We use p to denote a prime number, and write $p^s \| n$ when $p^s | n$ but $p^{s+1} \nmid n$. Finally, $\|x\|$ denotes $\min_{y\in\mathbb{Z}} |x-y|$.

In an effort to simplify our analysis, we adopt the following convention concerning the numbers ε and R. Whenever ε or R appear in a statement, either implicitly or explicitly, we assert that for each $\varepsilon > 0$, there exists a positive number $\eta_0(\varepsilon, s, k)$ such that the statement holds whenever $R = P^\eta$, with $0 < \eta \le \eta_0(\varepsilon, s, k)$. Note that the "value" of ε, and η_0, may change from statement to statement, and hence also the dependency of implicit constants on ε and η. Thus, for example, if $f \ll P^\varepsilon R^k$ and $g \ll P^\varepsilon R^{2k}$, then we shall conclude that $fg \ll P^\varepsilon$ without comment. Notice that since our iterative methods will involve only a finite number of statements (depending at most on k, s and ε), there is no danger of losing control of implicit constants through the successive changes implicit in our arguments. Finally, we use the symbol $\approx$ to indicate that constants and powers of R and P^ε are to be ignored.

For each $s \in \mathbb{N}$ we take $\phi_i = \phi_{i,s}$ $(i = 1, \dots, k)$ to be real numbers, with $0 \le \phi_i \le 1/k$, to be chosen later. We then take

$$P_j = 2^j P, \quad M_j = P^{\phi_j}, \quad H_j = P_j M_j^{-k}, \quad Q_j = P_j (M_1 \dots M_j)^{-1} \quad (1 \le j \le k).$$

For the sake of concision, we shall also adopt the convention of writing

$$\tilde{H}_j = \prod_{i=1}^{j} H_i \quad \text{and} \quad \tilde{M}_j = \prod_{i=1}^{j} M_i R \quad .$$

We define the modified forward difference operator, Δ_1^*, by

$$\Delta_1^* \left(f(x); h; m\right) = m^{-k} \left(f(x + hm^k) - f(x)\right),$$

and define Δ_j^* recursively by

$$\Delta_{j+1}^* \left(f(x); h_1, \dots, h_{j+1};\ m_1, \dots, m_{j+1}\right) = \Delta_1^* \left(\Delta_j^* \left(f(x); h_1, \dots, h_j; m_1, \dots, m_j\right); h_{j+1}; m_{j+1}\right).$$

We also adopt the convention that $\Delta_0^* \left(f(x); h; m\right) = f(x)$.

For $0 \le j \le k$ let

$$\Psi_j = \Psi_j(z; h_1, \dots, h_j; m_1, \dots, m_j) = \Delta_j^* \left(f(z); 2h_1, \dots, 2h_j; m_1, \dots, m_j\right)$$

where $f(z) = (z - h_1 m_1^k - \dots - h_j m_j^k)^k$.

Write

$$f_j(\alpha) = \sum_{x \in \mathcal{A}(Q_j, R)} e(\alpha x^k).$$

Also, write

$$F_j(\alpha) = \sum_{z, \mathbf{h}, \mathbf{m}} e\left(\alpha \Psi_j(z; \mathbf{h}; \mathbf{m})\right),$$

where the summation is over $z, \mathbf{h}, \mathbf{m}$ with

$$1 \le z \le P_j,\ M_i < m_i \le M_i R,\ m_i \in \mathcal{A}(P, R),\ 1 \le h_i \le 2^{j-i} H_i \quad (1 \le i \le j). \tag{2.1}$$

(Notice in particular the condition $m_i \in \mathcal{A}(P, R)$. In Wooley [**13**] the variables m_i were permitted to range over a complete interval, whereas the analyses of §§2 and 3 of that paper in fact allow the restriction to the set $\mathcal{A}(P, R)$).

We let $S_s^{(k)}(P, R)$ denote the number of solutions of the equation

$$x_1^k + \dots + x_s^k = y_1^k + \dots + y_s^k$$

with $x_i, y_i \in \mathcal{A}(P, R)$ $(1 \le i \le s)$. When no confusion is possible, we shall suppress the superscript k. Suppose that the real numbers λ_s and μ_s $(1 \le s < \infty)$ have the property that

$$S_s^{(k)}(P, R) \ll P^{\lambda_s + \varepsilon} \quad \text{and} \quad S_s^{(2k)}(P, R) \ll P^{\mu_s + \varepsilon} \quad . \tag{2.2}$$

Such numbers certainly exist, since we may trivially take $\lambda_s = 2s$ and $\mu_s = 2s$.

We list below some useful lemmata.

Lemma 2.1. *We have*

$$\int_0^1 \left|F_0(\alpha)^2 f_0(\alpha)^{2s}\right| d\alpha \ll P^\varepsilon M_1^{2s-1}\left(PM_1Q_1^{\lambda_s} + \int_0^1 \left|F_1(\alpha)f_1(\alpha)^{2s}\right| d\alpha\right). \quad (2.3)$$

Proof. This follows from Lemma 2.3 of Wooley [**13**], and the argument of the proof of Lemma 3.1 of Wooley [**13**], on considering the underlying diophantine equations.

We shall abbreviate an inequality of the form (2.3) symbolically by

$$F_0^2 f_0^{2s} \longmapsto F_1 f_1^{2s}.$$

Lemma 2.2. *Whenever* $0 < t < s$ *and* $1 \le j \le k-1$, *we have*

$$\int_0^1 \left|F_j(\alpha)f_j(\alpha)^{2s}\right| d\alpha \ll P^\varepsilon (Q_j^{\lambda_t})^{1/2}(\tilde{H}_j\tilde{M}_jM_{j+1}^{4s-2t-1}T_{j+1})^{1/2}, \quad (2.4)$$

where

$$T_{j+1} = T_{j+1}(P;\boldsymbol{\lambda};\boldsymbol{\phi}) = P\tilde{H}_j\tilde{M}_{j+1}Q_{j+1}^{\lambda_{2s-t}} + \int_0^1 \left|F_{j+1}(\alpha)f_{j+1}(\alpha)^{4s-2t}\right| d\alpha. \quad (2.5)$$

Proof. By Schwarz's inequality we have

$$\int_0^1 \left|F_j(\alpha)f_j(\alpha)^{2s}\right| d\alpha \ll \left(\int_0^1 |f_j(\alpha)|^{2t}\, d\alpha\right)^{1/2}\left(\int_0^1 \left|F_j(\alpha)^2 f_j(\alpha)^{4s-2t}\right| d\alpha\right)^{1/2}.$$

The proof of the lemma now follows by the arguments of the proofs of Lemmata 2.3 and 3.1 of Wooley [**13**], on considering the underlying diophantine equations.

We abbreviate an inequality of the form (2.4) symbolically by

$$\begin{array}{ccc} F_j f_j^{2s} & \longrightarrow & F_{j+1} f_{j+1}^{4s-2t} \\ & & \downarrow \\ & & f_j^{2t} \end{array}$$

There are two other ways of estimating the integral on the left hand side of equation (2.4).

(i) We may apply Hölder's inequality in the form

$$\int_0^1 \left|F_j(\alpha)f_j(\alpha)^{2s}\right| d\alpha \ll I_1^a I_2^b U_v^c U_w^d$$

where

$$I_m = \int_0^1 |F_j(\alpha)|^{2m}\, d\alpha \quad (m = 1, 2)$$

and

$$U_u = \int_0^1 |f_j(\alpha)|^{2u}\, d\alpha \quad (u = v, w),$$

in which v and w are non-negative integers and a, b, c, d are non-negative real numbers with

$$a + b + c + d = 1 \quad , \quad 2a + 4b = 1 \quad , \quad vc + wd = s \quad .$$

The second and fourth power mean values of F_j may be estimated in terms of the number of solutions of certain diophantine equations. Also, we have $U_v \ll Q_j^{\lambda_v + \varepsilon}$ and $U_w \ll Q_j^{\lambda_w + \varepsilon}$. We abbreviate an inequality *(H)* of this form symbolically by

$$F_j f_j^{2s} \Longrightarrow (F_j^2)^a (F_j^4)^b (f_j^{2v})^c (f_j^{2w})^d.$$

There is, of course, the possibility of using higher moments of $F_j(\alpha)$. However, estimates for such moments are too weak to be of value in the current state of knowledge.

(ii) We may apply the Hardy-Littlewood method along the lines of §3 of Vaughan [**8**]. We then abbreviate the resulting inequality *(M)* symbolically in the form

$$F_j f_j^{2s} \Longrightarrow (F_j)(f_j^{2s}).$$

By considering the underlying diophantine equations, we have

$$S_{s+1}(P, R) \ll \int_0^1 \left| F_0(\alpha)^2 f_0(\alpha)^{2s} \right| d\alpha,$$

and hence we may use a sequence Σ_s of connected inequalities (in the obvious sense) to bound $S_s(Q, R)$ in terms of $S_t(Q', R)$ $(t = 1, 2, \dots)$. Such a sequence will be called an *iterative procedure.* A finite subsequence of a sequence $(\Sigma_s)_1^\infty$ of iterative procedures will be called an *iterative scheme.*

Thus far, we have merely indicated possible methods for estimating certain integrals, without indicating how such estimates may be used to obtain upper bounds of the form (2.2) for $S_s^{(k)}(P, R)$. We now outline a possible strategy.

Suppose that we have taken $j + 1$ differences, and so are left to bound an expression of the form T_{j+1}, as defined by equation (2.5). By applying a process of the type *(H)* or *(M)*, we may obtain a bound of the form

$$T_{j+1} \ll P\tilde{H}_j \tilde{M}_{j+1} Q_{j+1}^{\lambda_{2s-t}} + V(P; \boldsymbol{\lambda}; \boldsymbol{\phi}), \tag{2.6}$$

for some expression $V(P; \boldsymbol{\lambda}; \boldsymbol{\phi})$ depending explicitly only on P, $\boldsymbol{\lambda}$, and $\boldsymbol{\phi} = (\phi_i)_{i=1}^{j+1}$. We may then obtain a bound for T_{j+1} by minimising the expression on the right-hand side of (2.6). In our applications, a close approximation to the minimum occurs when a choice of $\boldsymbol{\phi}$ is taken so that

$$P\tilde{H}_j \tilde{M}_{j+1} Q_{j+1}^{\lambda_{2s-t}} \approx V(P; \boldsymbol{\lambda}; \boldsymbol{\phi}).$$

This relation determines some equation,

$$\Lambda_{j+1}(\boldsymbol{\lambda};\boldsymbol{\phi})=0, \tag{2.7}$$

connecting the ϕ_i $(1\le i\le j+1)$ in an obvious manner.

With the optimal choice of $\boldsymbol{\phi}$ given by (2.7), the bound (2.4) now becomes

$$\int_0^1 \left|F_j(\alpha)f_j(\alpha)^{2s}\right| d\alpha \ll P^{\varepsilon}\left(P\tilde{H}_j^2\tilde{M}_j^2M_{j+1}^{4s-2t}Q_j^{\lambda_t}Q_{j+1}^{\lambda_{2s-t}}\right)^{1/2}.$$

This bound may now be used to bound an expression of the form T_j via Lemma 2.2, and we obtain an inequality of the form

$$T_j \ll P^{\varepsilon}\left(P\tilde{H}_{j-1}\tilde{M}_jQ_j^{\lambda_s}+\left(P\tilde{H}_j^2\tilde{M}_j^2M_{j+1}^{4s-2t}Q_j^{\lambda_t}Q_{j+1}^{\lambda_{2s-t}}\right)^{1/2}\right).$$

Optimising the right-hand side gives rise to a further equation connecting the $\boldsymbol{\phi}$, say $\Lambda_j(\boldsymbol{\lambda};\boldsymbol{\phi})=0$. We may continue this process, next bounding an expression of the form

$$\int_0^1 \left|F_{j-1}(\alpha)f_{j-1}(\alpha)^{2u}\right| d\alpha$$

in like manner, and so on.

In this way, for each s we obtain $j+1$ equations

$$\Lambda_i^{(s)}(\boldsymbol{\lambda};\boldsymbol{\phi})=0 \quad (1\le i\le j+1),$$

in $j+1$ variables ϕ_i $(1\le i\le j+1)$. These permit us to solve for $\boldsymbol{\phi}$ in terms of $\boldsymbol{\lambda}$, and provided that a solution is found with $0\le \phi_i\le 1/k$ for each $1\le i\le j+1$, then it follows that

$$\int_0^1 \left|F_0(\alpha)^2f_0(\alpha)^{2s}\right| d\alpha \ll P^{1+\varepsilon}M_1^{2s}Q_1^{\lambda_s},$$

with ϕ_1 given by the solution $\boldsymbol{\phi}$ of the simultaneous equations

$$\boldsymbol{\Lambda}^{(s)}(\boldsymbol{\lambda};\boldsymbol{\phi})=0.$$

It therefore follows that

$$S_{s+1}(P,R)\ll P^{\lambda'_{s+1}+\varepsilon},$$

with

$$\lambda'_{s+1}=\lambda_s(1-\phi_1)+1+2s\phi_1.$$

By adopting this entire process for $s=1,2,\dots$, we may define a new sequence of exponents, $\boldsymbol{\lambda}^+$, by taking

$$\lambda_s^+=\min\{\lambda'_s,\lambda_s\} \quad (s=1,2,\dots).$$

Further, we have the sequence of bounds

$$S_s(P,R) \ll P^{\lambda_s^+ + \varepsilon}.$$

In principle we may obtain the optimal $\boldsymbol{\lambda}$ by solving the equations $\boldsymbol{\lambda} = \boldsymbol{\lambda}^+$. Indeed, for smaller values of s, and in particular when the λ_t with $t > s$ do not occur explicitly in the formulae involving λ_s , this may be the easiest way to proceed. In practice, however, we proceed to calculate values for $\boldsymbol{\lambda}$ as follows. Starting from a known sequence $\boldsymbol{\lambda}$ we calculate $\boldsymbol{\lambda}^+$ as described above. Then we use the λ_s^+ in place of the λ_s in the equations $\boldsymbol{\Lambda}^{(r)}(\boldsymbol{\lambda};\boldsymbol{\phi}) = 0$. Thus, by applying this iterative scheme repeatedly, we obtain a sequence of sequences $(\lambda_s^{(r)})$ with $\lambda_s^{(r+1)} \le \lambda_s^{(r)}$ for each r and s. Since diagonal solutions provide us with the lower bound $\lambda_s^{(r)} \ge s$, the sequence must converge to some limit (λ_s^*). Moreover, $\boldsymbol{\lambda}^*$ has the property that

$$S_s(P,R) \ll P^{\lambda_s^* + \varepsilon}.$$

The method outlined above involves an iteration process in which each $\lambda_s^{(r+1)}$ $(1 \le s < \infty)$ depends on each $\lambda_s^{(r)}$ $(1 \le s < \infty)$. It will become plain that certain economies may be made in this procedure. Thus, for example, for s exceeding some s_0 we have $\lambda_s^* = 2s - k$. Further, for certain values of s the iterative procedure for λ_s may be independent of λ_t for $t > s$. In this latter case it may then be possible to obtain λ_s^* independently of λ_t^* $(t > s)$. In the sections which follow we discuss what were found to be the optimal methods for bounding the $\lambda_s^{(r)}$. In many instances the method is appropriate only for a single value of k. Nonetheless, for the purpose of more clearly indicating the recurring themes, we shall analyse the method as it applies more generally.

3. Estimates for the number of solutions of auxiliary equations

Our first step in facilitating the analysis outlined in the previous section will be to obtain estimates for the number of solutions of certain auxiliary equations, these enabling us to make use of the inequality

$$F_j f_j^{2s} \longrightarrow (F_j^2)^a (F_j^4)^b (f_j^{2v})^c (f_j^{2w})^d.$$

We first need to set up some notation.

Let us write ξ_i for $h_i m_i^k$. Then we have

$$\Psi_j = \Delta_j^* \left(f(z); 2h_1, \dots, 2h_j; m_1, \dots, m_j\right),$$

with

$$f(z) = (z - \xi_1 - \dots - \xi_j)^k.$$

Thus, in a manner similar to that of §2 of Vaughan [**8**], we obtain

$$\begin{aligned}\Psi_j &= \sum_{\theta_1=\pm 1}\cdots\sum_{\theta_j=\pm 1}\theta_1\dots\theta_j(m_1\dots m_j)^{-k}(z+\theta_1\xi_1+\dots+\theta_j\xi_j)^k\\ &= \sum_{\theta_1=\pm 1}\cdots\sum_{\theta_j=\pm 1}\sum_{\substack{u_0\ge 0\\ u_0+u_1+\dots+u_j=k}}\cdots\sum_{u_j\ge 0}\frac{k!\theta_1\dots\theta_j z^{u_0}(\theta_1\xi_1)^{u_1}\dots(\theta_j\xi_j)^{u_j}}{u_0!u_1!\dots u_j!(m_1\dots m_j)^k}\\ &= \sum_{\substack{u\ge 0\\ u+2v_1+\dots+2v_j=k-j}}\sum_{v_1\ge 0}\cdots\sum_{v_j\ge 0}\frac{k!2^j h_1\dots h_j z^u\xi_1^{2v_1}\dots\xi_j^{2v_j}}{u!(2v_1+1)!\dots(2v_j+1)!}\ .\end{aligned}$$

In particular, we obtain

$$\Psi_{k-2}=\frac{k!2^{k-3}}{3}h_1\dots h_{k-2}\left(3z^2+\sum_{i=1}^{k-2}\xi_i^2\right), \tag{3.1}$$

$$\Psi_{k-4}=\frac{k!2^{k-7}}{45}h_1\dots h_{k-4}\left(15z^4+30z^2\sum_{i=1}^{k-4}\xi_i^2+10\sum_{1\le i<j\le k-4}\xi_i^2\xi_j^2+3\sum_{i=1}^{k-4}\xi_i^4\right). \tag{3.2}$$

Let $R_j^{(s)}(P;\boldsymbol{\phi})$ denote the number of solutions of the diophantine equation

$$\sum_{i=1}^{s}\Psi_j\left(z_i;\mathbf{h}^{(i)};\mathbf{m}^{(i)}\right)=\sum_{i=1}^{s}\Psi_j\left(w_i;\mathbf{g}^{(i)};\mathbf{n}^{(i)}\right) \tag{3.3}$$

with

$$1\le z_i,w_i\le P_j \quad , \quad 1\le h_t^{(i)},g_t^{(i)}\le 2^{j-t}H_t \quad , \tag{3.4}$$

$$M_t<m_t^{(i)},n_t^{(i)}\le M_tR \quad , \quad m_t^{(i)},n_t^{(i)}\in\mathcal{A}(P,R) \quad , \tag{3.5}$$

for $1\le t\le j$, $1\le i\le s$.

We shall be concerned only with estimates for $R_j^{(s)}$ with $s=1$ or 2, the estimates obtainable by current methods being otherwise too weak to be of value. We begin by establishing a relation between $R_j^{(2)}$, and $R_j^{(1)}$ and $R_{j+1}^{(1)}$.

Lemma 3.1. *When $1\le j\le k-2$, we have*

$$R_j^{(2)}(P;\phi_1,\dots,\phi_j)\ll P\tilde{H}_j^2\tilde{M}_j^2R_j^{(1)}(P;\phi_1,\dots,\phi_j)+\tilde{H}_j^2\tilde{M}_j^2R_{j+1}^{(1)}(P;\phi_1,\dots,\phi_j,0).$$

Proof. On considering the underlying diophantine equation, by (3.3) we have

$$R_j^{(2)}(P;\boldsymbol{\phi})=\int_0^1|F_j(\alpha)|^4\,d\alpha. \tag{3.6}$$

But by applying standard Weyl differencing, combined with Cauchy's inequality, we have

$$|F_j(\alpha)|^2 \ll P\tilde{H}_j^2\tilde{M}_j^2 + \tilde{H}_j\tilde{M}_j\,|G(\alpha)|\,,$$

where

$$G(\alpha) = \sum_{\mathbf{h},\mathbf{m}}\ \sum_{1\le h\le P_j}\ \sum_{1\le z\le P_j-h} e\left(\alpha\left(\Psi_j(z+h;\mathbf{h};\mathbf{m}) - \Psi_j(z;\mathbf{h};\mathbf{m})\right)\right),$$

and the summation over $\mathbf{h}$ and $\mathbf{m}$ is over the ranges given in (3.4) and (3.5). Then from (3.6) we have

$$R_j^{(2)}(P;\boldsymbol{\phi}) \ll P\tilde{H}_j^2\tilde{M}_j^2\int_0^1 |F_j(\alpha)|^2\,d\alpha + \tilde{H}_j\tilde{M}_j\int_0^1 \left|G(\alpha)F_j(\alpha)^2\right| d\alpha.$$

Then by applying Schwarz's inequality, and considering the underlying diophantine equations, we have

$$R_j^{(2)}(P;\boldsymbol{\phi}) \ll P\tilde{H}_j^2\tilde{M}_j^2R_j^{(1)}(P;\boldsymbol{\phi}) + \tilde{H}_j\tilde{M}_j\left(R_j^{(2)}(P;\boldsymbol{\phi})\cdot S\right)^{1/2},$$

where S denotes the number of solutions of the equation

$$\Delta_1^*\left(\Psi_j(z;\mathbf{h};\mathbf{m});h;1\right) = \Delta_1^*\left(\Psi_j(w;\mathbf{g};\mathbf{n});g;1\right),$$

with the variables $\mathbf{h},\mathbf{g},\mathbf{m},\mathbf{n}$, satisfying (3.4) and (3.5), and with $1\le h,g\le P_j$, $1\le z\le P_j-h$ and $1\le w\le P_j-g$. But we have

$$\begin{aligned}2^k\Delta_1^*\left(\Psi_j(z;\mathbf{h};\mathbf{m});h;1\right) &= \Delta_{j+1}^*\left((2z-2\xi_1-\cdots-2\xi_j)^k;4\mathbf{h},2h;\mathbf{m},1\right)\\ &= \Psi_{j+1}(2z+h;2\mathbf{h},h;\mathbf{m},1),\end{aligned}$$

and hence the result follows on noting that $2z+h<2P_j=P_{j+1}$.

Next we provide an estimate for $R_j^{(1)}$ which is valid uniformly in k and j. Later we shall refine this estimate for a fairly large set of k and j.

Lemma 3.2. *When $1\le j\le k-2$, we have*

$$R_j^{(1)}(P;\boldsymbol{\phi}) \ll P^{1+\varepsilon}\tilde{H}_j\tilde{M}_j^2.$$

Proof. We have

$$R_j^{(1)}(P;\boldsymbol{\phi}) = \sum_n\left(\sum_{\mathbf{h}} R(n;\mathbf{h})\right)^2,$$

where the second summation is over $\mathbf{h}$ satisfying (3.4), and where for a fixed $\mathbf{h}$, $R(n;\mathbf{h})$ denotes the number of solutions of the equation $\Psi_j(z;\mathbf{h};\mathbf{m})=n$ with z

and $\mathbf{m}$ satisfying (3.4) and (3.5). But if $z, \mathbf{h}$ and $\mathbf{m}$ satisfy (3.4) and (3.5), then $\Psi_j(z;\mathbf{h};\mathbf{m})$ is divisible by $h_1 \dots h_j$, and further is non-zero. Therefore

$$R_j^{(1)}(P;\boldsymbol{\phi}) \le \sum_n \left(\sum_{h_1|n} \cdots \sum_{h_j|n} R(n;\mathbf{h}) \right)^2 .$$

But $R(0;\mathbf{h}) = 0$, so on combining standard estimates for the divisor function with Cauchy's inequality, we obtain

$$R_j^{(1)}(P;\boldsymbol{\phi}) \ll P^{\varepsilon} \sum_{\mathbf{h}} \sum_n R(n;\mathbf{h})^2. \tag{3.7}$$

Further, by assigning values to the $\mathbf{m}$, and solving directly for z, we have $R(n;\mathbf{h}) \ll \tilde{M}_j$, and hence the desired conclusion follows from (3.7).

Before we consider refinements of the above lemma, we require a definition. When $k-j$ is odd, or when $k-j = 2$ or 4, we put $J = \left[\frac{k-j}{2}\right]$. We then define $K_j(P;\boldsymbol{\phi})$ to be the number of solutions of the system of diophantine equations

$$\sum_{i=1}^{j} h_i^{2r}(m_i^{2rk} - n_i^{2rk}) = 0 \quad (1 \le r \le J) \tag{3.8}$$

with $\mathbf{h}, \mathbf{m}$ and $\mathbf{n}$ satisfying (3.4) and (3.5). Notice, in particular, that by counting diagonal solutions of (3.8), we have

$$K_j(P;\boldsymbol{\phi}) \gg \tilde{H}_j \tilde{M}_j. \tag{3.9}$$

We now establish a reduction formula relating $R_j^{(1)}$ with K_j.

Lemma 3.3. *Suppose that $1 \le j \le k-2$, and $k-j$ is odd, or $k-j = 2$ or 4. Then*

$$R_j^{(1)}(P;\boldsymbol{\phi}) \ll P^{1+\varepsilon} K_j(P;\boldsymbol{\phi}).$$

Proof. In each of the cases under consideration, we may start by observing that $h_1 \dots h_j$ divides $\Psi_j(z;\mathbf{h};\mathbf{m})$, and so as in the proof of Lemma 3.2 we have

$$R_j^{(1)}(P;\boldsymbol{\phi}) \ll P^{\varepsilon} R^*(P;\boldsymbol{\phi}),$$

where now we write $R^*(P;\boldsymbol{\phi})$ for the number of solutions of the equation

$$\Psi_j(z;\mathbf{h};\mathbf{m}) = \Psi_j(w;\mathbf{h};\mathbf{n}) \tag{3.10}$$

with $z, w, \mathbf{h}, \mathbf{m}, \mathbf{n}$ satisfying (3.4) and (3.5).

We now divide into cases.

(i) $k-j=2$. Then from (3.1), the equation (3.10) in this case becomes

$$3(z^2-w^2)+\sum_{i=1}^{k-2}h_i^2(m_i^{2k}-n_i^{2k})=0. \tag{3.11}$$

From (3.8), the number of solutions with $z=w$ is

$$\ll PK_{k-2}(P;\boldsymbol{\phi}). \tag{3.12}$$

Now count solutions of (3.11) with $z\neq w$. We may assign $\mathbf{h}$, $\mathbf{m}$ and $\mathbf{n}$ in $O(\tilde{H}_{k-2}\tilde{M}_{k-2}^2)$ ways. Fixing this choice, we may use standard estimates for the divisor function to deduce that there are $O(P^\varepsilon)$ solutions of this type in z and w. Then the total number of solutions of this type is

$$\ll P^\varepsilon\tilde{H}_{k-2}\tilde{M}_{k-2}^2\ll P^{1+\varepsilon}\tilde{H}_{k-2}\tilde{M}_{k-2}. \tag{3.13}$$

When $k-j=2$, the result now follows on combining (3.9), (3.12) and (3.13).

(ii) $k-j=4$. Then from (3.2), the equation (3.10) in this case becomes

$$15(u^2-v^2)=10(\Xi_1^2-\Gamma_1^2)+2(\Xi_2-\Gamma_2), \tag{3.14}$$

in which

$$u=z^2+\Xi_1,\quad \Xi_1=\sum_{i=1}^{k-4}h_i^2m_i^{2k},\quad \Xi_2=\sum_{i=1}^{k-4}h_i^4m_i^{4k},$$

and v,Γ_1,Γ_2 are defined similarly in terms of w,$\mathbf{h}$ and $\mathbf{n}$.

Consider first solutions of (3.14) counted by $R^*(P;\boldsymbol{\phi})$ with $u\neq v$. We may assign $\mathbf{h},\mathbf{m}$ and $\mathbf{n}$ in $O(\tilde{H}_{k-4}\tilde{M}_{k-4}^2)$ ways. Fixing this choice, we may then use standard estimates for the divisor function to deduce that there are $O(P^\varepsilon)$ solutions of this type in u and v, and hence in z and w. Then the total number of solutions of this type is

$$\ll P^\varepsilon\tilde{H}_{k-4}\tilde{M}_{k-4}^2\ll P^{1+\varepsilon}\tilde{H}_{k-4}\tilde{M}_{k-4}. \tag{3.15}$$

Now consider solutions of (3.14) counted by $R^*(P;\boldsymbol{\phi})$ with $u=v$. Then we have

$$(z^2-w^2)+\sum_{i=1}^{k-4}h_i^2(m_i^{2k}-n_i^{2k})=0. \tag{3.16}$$

As in case (i), the number of solutions with $z\neq w$ is

$$\ll P^{1+\varepsilon}\tilde{H}_{k-4}\tilde{M}_{k-4}. \tag{3.17}$$

Otherwise $z=w$, and from (3.16) we have $\Xi_1=\Gamma_1$, and hence from (3.14), $\Xi_2=\Gamma_2$. Then from (3.8), the total number of solutions of this type is

$$\ll PK_{k-4}(P;\boldsymbol{\phi}). \tag{3.18}$$

Then when $k-j=4$, the result follows by combining (3.9), (3.15), (3.17) and (3.18).

(iii) $k-j$ odd. Write $k-j=2J+1$. Then

$$\Psi_j(z;\mathbf{h};\mathbf{m}) = Ch_1\dots h_j z\left(\sum_{r=0}^{J} c_r z^{2r}\right) \tag{3.19}$$

where C depends at most on k and j, and $c_r = c_r(\boldsymbol{\xi})$ $(0\le r\le J)$ is a symmetric polynomial in $\xi_1^2,\dots,\xi_j^2$ of degree $J-r$, with coefficients depending at most on k and j.

On noting that $\Psi_j(z;\mathbf{h};\mathbf{m})$ is divisible by $z,h_1,\dots,h_j$, we find, as in the proof of Lemma 3.2, that

$$R_j^{(1)}(P;\boldsymbol{\phi}) \ll P^{\varepsilon}R^{+}(P;\boldsymbol{\phi}),$$

where now we write $R^{+}(P;\boldsymbol{\phi})$ for the number of solutions of the equation

$$\Psi_j(z;\mathbf{h};\mathbf{m}) = \Psi_j(z;\mathbf{h};\mathbf{n}), \tag{3.20}$$

with $z,\mathbf{h},\mathbf{m},\mathbf{n}$ satisfying (3.4) and (3.5). But on noting (3.19), equation (3.20) becomes

$$\sum_{r=0}^{J}\left(c_r(h_1m_1^k,\dots,h_jm_j^k) - c_r(h_1n_1^k,\dots,h_jn_j^k)\right)z^{2r} = 0. \tag{3.21}$$

Consider first solutions of (3.21) with

$$c_r(h_1m_1^k,\dots,h_jm_j^k) \ne c_r(h_1n_1^k,\dots,h_jn_j^k)$$

for some r. We may assign $\mathbf{h},\mathbf{m}$ and $\mathbf{n}$ in $O(\tilde{H}_j\tilde{M}_j^2)$ ways. Fixing this choice, we have that z is determined by a non-trivial polynomial. So there are $O(1)$ such solutions in z, and hence the number of solutions of this type is

$$\ll \tilde{H}_j\tilde{M}_j^2 \ll P\tilde{H}_j\tilde{M}_j. \tag{3.22}$$

Otherwise

$$c_r(h_1m_1^k,\dots,h_jm_j^k) = c_r(h_1n_1^k,\dots,h_jn_j^k)$$

for $0\le r\le J$. But then, by using elementary results on symmetric polynomials, we have

$$\sum_{i=1}^{j} h_i^{2r}(m_i^{2rk}-n_i^{2rk}) = 0 \quad (1\le r\le J).$$

Then from (3.8), the number of solutions of this type is

$$\ll PK_j(P;\boldsymbol{\phi}). \tag{3.23}$$

When $k-j$ is odd, the result now follows on combining (3.9), (3.22) and (3.23), and this completes the proof of the lemma.

We must now attend to the matter of bounding $K_j(P;\boldsymbol{\phi})$. We might hope to achieve the essentially best possible bound $K_j(P;\boldsymbol{\phi}) \ll P^\varepsilon \tilde{M}_j \tilde{H}_j$, dominated by diagonal solutions. In the light of our estimates for $S_s(P,R)$, this may seem excessively optimistic, yet we very nearly achieve this goal. Unfortunately our methods are somewhat diverse, and will take a little time to explain. More precise estimates can be obtained by our methods, but we choose simplicity of exposition. We start with a useful lemma, depending for its effectiveness on estimates for the number of solutions of a homogeneous system of equations.

We define $S_s(Q,R;t,k)$ to be the number of solutions of the system of diophantine equations

$$\sum_{i=1}^{s}(x_i^{2nk}-y_i^{2nk})=0 \quad (1\le n\le t)$$

with $x_i,y_i\in\mathcal{A}(Q,R)$ $(1\le i\le s)$. We note that estimates for $S_s(Q,R;t,k)$ are available from Wooley [**14**, **15**].

Lemma 3.4. *Suppose that* $2\le j\le k-2$. *Let* $l=[j/2]$, *and define*

$$L_{i,r}(P)=H_i^{-1}S_{2r}(M_iR,R;J,k)+\big(S_r(M_iR,R;J,k)\big)^2,$$

and

$$L_{i,l}^*(P)=\begin{cases} L_{i,l}(P) & j \text{ even},\\ (L_{i,l}(P)L_{i,l+1}(P))^{1/2} & j \text{ odd}.\end{cases}$$

Then

$$K_j(P;\boldsymbol{\phi})\ll P^\varepsilon\tilde{H}_j\left(\prod_{i=1}^{j}\min\left\{L_{i,l}^*(P)\ ,\ S_j(M_iR,R;J,k)\right\}\right)^{1/j}.$$

Proof. Write

$$g_r(\boldsymbol{\alpha};H,Q,R)=\sum_{1\le h\le H}\left|\sum_{x\in\mathcal{A}(Q,R)}e(\alpha_Jh^{2J}x^{2Jk}+\cdots+\alpha_1h^2x^{2k})\right|^r.$$

Then we have

$$K_j(P;\boldsymbol{\phi})\ll\int_{\mathbb{T}^J}\prod_{i=1}^{j}g_2(\boldsymbol{\alpha};2^{j-i}H_i,M_iR,R)d\boldsymbol{\alpha}, \tag{3.24}$$

where here, and throughout, we write $\mathbb{T}$ for $[0,1]$.

As applications of Hölder's inequality, we have

$$\begin{aligned} g_2(\boldsymbol{\alpha};H,Q,R)^j &\ll H^{j-1}g_{2j}(\boldsymbol{\alpha};H,Q,R),\\ g_2(\boldsymbol{\alpha};H,Q,R)^j &\ll H^{j-2}g_j(\boldsymbol{\alpha};H,Q,R)^2,\\ g_2(\boldsymbol{\alpha};H,Q,R)^j &\ll H^{j-2}g_{j-1}(\boldsymbol{\alpha};H,Q,R)g_{j+1}(\boldsymbol{\alpha};H,Q,R). \end{aligned}$$

But by considering the underlying diophantine equations, we have

$$\int_{\mathbb{T}^J} g_{2j}(\boldsymbol{\alpha};H,Q,R)d\boldsymbol{\alpha} \ll HS_j(Q,R;J,k). \tag{3.25}$$

Also, for each integer r we have that

$$\int_{\mathbb{T}^J} g_{2r}(\boldsymbol{\alpha};H,Q,R)^2 d\boldsymbol{\alpha}$$

is bounded above by the number of solutions of the system of diophantine equations

$$h^{2n}\sum_{i=1}^{r}(x_i^{2nk}-y_i^{2nk}) = g^{2n}\sum_{i=1}^{r}(u_i^{2nk}-v_i^{2nk}) \quad (1\le n\le J), \tag{3.26}$$

with $1 \le h, g \le H$ and $x_i, y_i, u_i, v_i \in \mathcal{A}(Q,R)$. The number of solutions counted in which the left hand side of (3.26) is zero is

$$\ll (HS_r(Q,R;J,k))^2\,.$$

Meanwhile, if the left hand side is non-zero, using a by now familiar argument, we may bound the number of solutions of (3.26) by $P^\varepsilon H^{1+\varepsilon}$ times the number of solutions of the system

$$\sum_{i=1}^{r}(x_i^{2nk}-y_i^{2nk}) = \sum_{i=1}^{r}(u_i^{2nk}-v_i^{2nk}) \quad (1\le n\le J),$$

with $x_i, y_i, u_i, v_i \in \mathcal{A}(Q,R)$. Since this is $\ll S_{2r}(Q,R;J,k)$, we have

$$\int_{\mathbb{T}^J} g_{2r}(\boldsymbol{\alpha};2^{j-i}H_i,M_iR,R)^2 d\boldsymbol{\alpha} \ll P^\varepsilon H_i^2 L_{i,r}(P). \tag{3.27}$$

Furthermore, by using Schwarz's inequality combined with the analysis above, we deduce that when u is an odd integer, we have

$$\int_{\mathbb{T}^J} g_{u-1}(\boldsymbol{\alpha};H_i,M_iR,R)g_{u+1}(\boldsymbol{\alpha};H_i,M_iR,R)d\boldsymbol{\alpha} \ll P^\varepsilon H_i^2\left(L_{i,u-1}(P)L_{i,u+1}(P)\right)^{1/2}. \tag{3.28}$$

Now applying Hölder's inequality to (3.24), we may combine (3.25), (3.27) and (3.28) to complete the proof of the lemma.

Before describing our final approach to bounding K_j, we shall require an elementary lemma on solutions of binary quadratic forms.

Lemma 3.5. *The number of solutions, $S(a,b,c;P)$, of the equation*

$$ax^2+by^2=c \quad (abc\neq 0)$$

with $1\le x,y\le P$ is $\ll (abcP)^{\varepsilon}$.

Proof. The conclusion of the lemma follows in an elementary manner from results of Chapter 11 of Hua [**2**]. We shall therefore merely sketch the required argument.

We first note that by changes of variable, combined with standard estimates for the divisor function, it suffices to show that when d is a non-zero square-free number, then the number of solutions of the equation

$$X^2-dY^2=n \quad (n\neq 0) \tag{3.29}$$

with $(X,Y)=1$ and $1\le X,Y\le P$ is $O\left((ndP)^{\varepsilon}\right)$. By Theorem 4.1 of Hua [**2**], for each solution (X,Y) of (3.29), there exists a unique integer l, with $0\le l<2n$, satisfying $l^2\equiv 4d \pmod{4n}$. Since d is square-free, the number of solutions of this congruence is $O(n^{\varepsilon})$, and so it suffices to show that there are $O((ndP)^{\varepsilon})$ solutions of (3.29) corresponding to each l.

(i) Suppose that $d<0$. By Theorem 4.3 of Hua [**2**], there are at most 4 solutions (X,Y) of (3.29) corresponding to each l.

(ii) Suppose that $d>0$. Then it follows from Theorems 4.2 and 4.4 of Hua [**2**] that, if (X,Y) and (X',Y') are any two solutions of (3.29) corresponding to the same l, then

$$X+\sqrt{d}Y=\pm(t+u\sqrt{d})^k(X'+\sqrt{d}Y'), \tag{3.30}$$

for some integer k, and choice of $+$ or $-$. Here (t,u) is the unique integer solution of the equation $t^2-du^2=1$ with $t>0, u>0$, and $t+u\sqrt{d}$ least. But for each solution of (3.29) we have $1\le |X+\sqrt{d}Y|\le (1+\sqrt{d})P$, and hence the desired conclusion follows from (3.30) on noting that $t+u\sqrt{d}\ge 1+\sqrt{d}\ge 2$.

This completes the proof of the lemma.

We now aim to exploit the differing sizes of the H_i via the previous lemma. We shall consider the number of solutions, $N_j(P;\boldsymbol{\phi})$, of the equation

$$\sum_{i=1}^{j} h_i^2(m_i^{2k}-n_i^{2k})=0, \tag{3.31}$$

with $\mathbf{h},\mathbf{m}$ and $\mathbf{n}$ satisfying (3.4) and (3.5). First, however, we shall consider the number of solutions, $N_j^*(P;\boldsymbol{\phi})$, of the equation (3.31) subject to the additional condition $m_i\neq n_i$ $(1\le i\le j)$.

We suppose in the following four lemmata that $j\ge 1$ and $\phi_1\ge\phi_2\ge\dots\ge\phi_j$, as is the case in our applications.

Lemma 3.6. *We have*

$$N_j^*(P;\boldsymbol{\phi}) \ll P^\varepsilon \tilde{H}_j \tilde{M}_j \left(1 + \sum_{\substack{i=3 \\ j-i \text{ even}}}^{j} \frac{\tilde{M}_i}{H_{i-1}H_i} \right).$$

Proof. We proceed by induction on j. When $j = 1$ the estimate is trivial, and when $j = 2$ the estimate follows almost trivially by use of divisor function estimates. Further, we have that $N_3^*(P;\boldsymbol{\phi})$ is the number of solutions of the equation

$$h_1^2(n_1^{2k} - m_1^{2k}) + h_2^2(n_2^{2k} - m_2^{2k}) = h_3^2(m_3^{2k} - n_3^{2k}) \neq 0$$

with $\mathbf{h}, \mathbf{m}, \mathbf{n}$ satisfying (3.4) and (3.5). Thus, by standard estimates for the divisor function, we have

$$N_3^*(P;\boldsymbol{\phi}) \ll P^\varepsilon \tilde{H}_2 \tilde{M}_2^2.$$

Therefore, recalling the condition on $\boldsymbol{\phi}$, and applying the trivial inequality

$$|z_1 \dots z_n| \leq |z_1|^n + \dots + |z_n|^n,$$

we obtain

$$N_3^*(P;\boldsymbol{\phi}) \ll P^\varepsilon \tilde{H}_3^{2/3} \tilde{M}_3^{4/3} \ll P^\varepsilon \left(\tilde{H}_3 \tilde{M}_3 + \tilde{M}_3^2 \right),$$

and so the result follows when $j = 3$.

Suppose now that $j > 3$. By applying Lemma 3.5, we deduce that the number of solutions of (3.31) counted by $N_j^*(P;\boldsymbol{\phi})$ with

$$\sum_{i=1}^{j-2} h_i^2(m_i^{2k} - n_i^{2k}) \neq 0 \tag{3.32}$$

is

$$\ll P^\varepsilon \tilde{H}_{j-2} \tilde{M}_j^2 = P^\varepsilon \tilde{H}_j \tilde{M}_j \left(\frac{\tilde{M}_j}{H_{j-1}H_j} \right). \tag{3.33}$$

Meanwhile, by the inductive hypothesis, the number of solutions of (3.31) counted by $N_j^*(P;\boldsymbol{\phi})$ with the left hand side of (3.32) zero is

$$\begin{aligned} &\ll P^\varepsilon H_j M_j^2 \tilde{H}_{j-2} \tilde{M}_{j-2} \left(1 + \sum_{\substack{i=3 \\ j-i \text{ even}}}^{j-2} \frac{\tilde{M}_i}{H_{i-1}H_i} \right) \\ &\ll P^\varepsilon \tilde{H}_j \tilde{M}_j \left(1 + \sum_{\substack{i=3 \\ j-i \text{ even}}}^{j-2} \frac{\tilde{M}_i}{H_{i-1}H_i} \right). \end{aligned} \tag{3.34}$$

The proof of the lemma is now completed on combining (3.33) and (3.34).

Lemma 3.7. *We have*

$$N_j(P;\boldsymbol{\phi}) \ll P^\varepsilon \tilde{H}_j \tilde{M}_j \left(1 + \sum_{i=3}^{j} \frac{\tilde{M}_i}{H_{i-1}H_i}\right).$$

Proof. Let $t \ge 0$ and i_u $(1 \le u \le t)$ be integers with $1 \le i_1 < i_2 < \cdots < i_t \le j$. Now consider the number of solutions of (3.31) counted by $N_j(P;\boldsymbol{\phi})$ in which $m_i \ne n_i$ whenever $i = i_u$ $(1 \le u \le t)$, and $m_i = n_i$ otherwise. On noting that when $t = 0$ there are only diagonal solutions, we deduce from Lemma 3.6, by a change of variables, that the number of such solutions is

$$\ll P^\varepsilon \tilde{H}_j \tilde{M}_j \left(1 + \sum_{u=3}^{t} \frac{\prod_{v=1}^{u} M_{i_v} R}{H_{i_{u-1}} H_{i_u}}\right).$$

The lemma now follows on observing that $M_{i_u}/H_{i_u} \ll M_{i_{u-1}+1}/H_{i_{u-1}+1}$.

There are a number of improvements which are of use in special circumstances.

Lemma 3.8. *We have*

$$N_j(P;\boldsymbol{\phi}) \ll P^\varepsilon \left(H_j M_j N_{j-1}(P;\boldsymbol{\phi}) + \tilde{H}_{j-1}\tilde{M}_{j-1}^2\right),$$

and in particular

$$N_3(P;\boldsymbol{\phi}) \ll P^\varepsilon \tilde{H}_3 \tilde{M}_3 \left(1 + \frac{\tilde{M}_2}{H_3 M_3}\right).$$

Proof. The number of solutions of (3.31) counted by $N_j(P;\boldsymbol{\phi})$ with $m_j = n_j$ is $\ll H_j M_j R N_{j-1}(P;\boldsymbol{\phi})$. Meanwhile, by using standard estimates for the divisor function, the number of solutions with $m_j \ne n_j$ is $\ll P^\varepsilon \tilde{H}_{j-1}\tilde{M}_{j-1}^2$.

The bound for $N_3(P;\boldsymbol{\phi})$ given by Lemma 3.8 is superior to that of Lemma 3.7 whenever $H_2 < M_3^2$.

Lemma 3.9. *When $j \ge 2$ we have*

$$N_j(P;\boldsymbol{\phi}) \ll P^\varepsilon \left(H_{j-1}H_j M_{j-1} M_j N_{j-2}(P;\boldsymbol{\phi}) + \tilde{H}_{j-2}\tilde{M}_{j-2}^2 (H_j M_j + M_{j-1}^2 M_j^2)\right),$$

and in particular

$$N_4(P;\boldsymbol{\phi}) \ll P^\varepsilon \tilde{H}_4 \tilde{M}_4 \left(1 + \frac{\tilde{M}_2}{H_3 M_3} + \frac{\tilde{M}_4}{H_3 H_4}\right).$$

Proof. The number of solutions of (3.31) counted by $N_j(P;\boldsymbol{\phi})$ with

$$h_{j-1}^2(m_{j-1}^{2k} - n_{j-1}^{2k}) + h_j^2(m_j^{2k} - n_j^{2k}) = 0 \qquad (3.35)$$

is

$$\ll P^{\varepsilon} H_{j-1} H_j M_{j-1} M_j N_{j-2}(P; \boldsymbol{\phi}).$$

Meanwhile, if the left hand side of (3.35) is non-zero, we may either apply standard divisor function estimates, or Lemma 3.5. Thus, the number of solutions in this case with $m_i \neq n_i$ ($i = j-1$ or $i = j$) is

$$\ll P^{\varepsilon} \tilde{H}_{j-2} \tilde{M}_{j-2}^2 (H_j M_j + H_{j-1} M_{j-1} + M_{j-1}^2 M_j^2).$$

This completes the proof of the lemma.

We now collect together the conclusions of this section in a simplified form, this being of use in our later applications.

Theorem 3.10. *Suppose that $1 \leq j \leq k-2$. Let $l = [j/2]$, $J = \left[\frac{k-j}{2}\right]$, and $\delta_r = \lambda_r^{(2Jk)} - r$ $(r \geq 1)$. Suppose that δ_r is increasing with r, and let e be 0 or 1 according as j is even or odd.*

(Ia) Unconditionally, if $j = 1$, or

(Ib) if $k-j$ is odd, or $k-j = 2$ or 4, and any one of the following conditions hold,

(i) $1 \leq j \leq J+1$;

(ii) $2 + e \leq j \leq 2J + 2 - e$ and $(k + \delta_{j+e})\phi_1 \leq 1$;

(iii) when $j \geq 3$, we have

$$\sum_{i=1}^{I} \phi_i + k(\phi_{I-1} + \phi_I) \leq 2 \quad (3 \leq I \leq j);$$

then

$$\int_0^1 |F_j(\alpha)|^2 d\alpha \ll P^{1+\varepsilon} \tilde{M}_j \tilde{H}_j. \tag{a}$$

If none of (i)-(iii) hold, we have

$$\int_0^1 |F_j(\alpha)|^2 d\alpha \ll P^{1+\varepsilon} \tilde{M}_j^{1+\sigma} \tilde{H}_j, \tag{b}$$

where $\sigma = \delta_j / j$. Furthermore, if $(k + \delta_{2(l+f)} - 2\delta_{l+f})\phi_1 \leq 1$ $(f = 0, e)$, we may take $\sigma = (\delta_l + \delta_{l+e})/j$.

(II) In any case, we have

$$\int_0^1 |F_j(\alpha)|^2 d\alpha \ll P^{1+\varepsilon} \tilde{M}_j^2 \tilde{H}_j. \tag{c}$$

Proof. Part (Ia) follows from Lemma 2.1 of Vaughan [**9**], and Part (II) follows from Lemma 3.2, on considering the underlying diophantine equation. So suppose that

$k-j$ is odd, or $k-j=2$ or 4. Then estimate (a) will follow from Lemma 3.3, on considering the underlying diophantine equation, providing we can show that

$$K_j(P;\boldsymbol{\phi}) \ll P^{\varepsilon}\tilde{H}_j\tilde{M}_j. \tag{3.36}$$

The number of solutions of the system of equations

$$\sum_{i=1}^{s}(x_i^{2nk}-y_i^{2nk})=0 \quad (1\le n\le t)$$

with $1\le x_i,y_i\le P$ $(1\le i\le s)$ is $O(P^s)$ when $1\le s\le t$. This follows by an elimination argument, for example. Also, when $s=t+1$, the number of solutions is $O(P^{t+1+\varepsilon})$, by Theorem 1 of Wooley [**14**]. Then when (i) holds, we plainly have

$$S_j(M_iR,R;J,k)\ll (M_iR)^{j+\varepsilon},$$

and hence (3.36) follows, by Lemma 3.4.

Now suppose that condition (ii) holds. Then we have $l+e\le J+1$, so as above,

$$S_r(M_iR,R;J,k)\ll (M_iR)^{r+\varepsilon}$$

when $r=l,l+e$. Now, by discarding all but one of the implicit equations, we deduce that for each u,

$$S_u(M_iR,R;J,k)\ll S_u^{(2Jk)}(M_iR,R)\ll (M_iR)^{\lambda_u^{(2Jk)}+\varepsilon}. \tag{3.37}$$

Hence, by the definition of H_i, the condition on ϕ_1, and the (implicit) assumption $\phi_1\ge\phi_i$ $(i\ge 1)$, we have

$$H_i^{-1}S_{2r}(M_iR,R;J,k)\ll P^{\varepsilon}M_i^{2r}$$

when $r=l,l+e$. Then, in Lemma 3.4, we have $L_{i,l}^*(P)\ll P^{\varepsilon}M_i^{j}$, and once again (3.36) follows.

Now suppose that condition (iii) holds. Then by Lemma 3.7, we have

$$N_j(P;\boldsymbol{\phi})\ll P^{\varepsilon}\tilde{H}_j\tilde{M}_j,$$

whence, by discarding all but one of the subsistent equations, (3.36) follows once again.

Finally, if none of (i)-(iii) hold, we use (3.37) in Lemma 3.4 with $u=l+f,j+f$ $(f=0,e)$ to obtain estimate (b).

This completes the proof of the theorem.

Theorem 3.11. *Suppose that* $1 \leq j \leq k-3$*, and let* $J = \left[\frac{k-j-1}{2}\right]$*. Otherwise make the same hypotheses, and adopt the same notation, as in Theorem* 3.10*.*

(I) Suppose that $3 \leq k-j \leq 5$ *or, when* $j=1$ *and* $k \geq 9$*, that* k *is odd. Then if* $j=1$*, or any one of conditions (i)-(iii) of Theorem* 3.10 *hold, then*

$$\int_0^1 |F_j(\alpha)|^4 d\alpha \ll P^{2+\varepsilon} \tilde{M}_j^3 \tilde{H}_j^3. \tag{a}$$

If none of the conditions (i)-(iii) of Theorem 3.10 *hold, then*

$$\int_0^1 |F_j(\alpha)|^4 d\alpha \ll P^{2+\varepsilon} \tilde{M}_j^{3+\sigma} \tilde{H}_j^3. \tag{b}$$

(II) In any case, we have

$$\int_0^1 |F_j(\alpha)|^4 d\alpha \ll P^{2+\varepsilon} \tilde{M}_j^4 \tilde{H}_j^3. \tag{c}$$

Proof. When $j=1$, part (I) follows from equations (2.14) and (2.15) of Vaughan [**9**]. Next, note that by Lemma 3.2,

$$R_{j+1}^{(1)}(P;\phi_1,\ldots,\phi_j,0) \ll P^{2+\varepsilon}\tilde{H}_j\tilde{M}_j^2,$$

and hence part (II) follows from Lemmata 3.1 and 3.2. So suppose that $3 \leq k-j \leq 5$. When one of conditions (i)-(iii) hold, estimate (a) will follow from Lemmata 3.1 and 3.3, on considering the underlying diophantine equation, providing we can show that

$$K_j(P;\boldsymbol{\phi}) \ll P^{\varepsilon}\tilde{H}_j\tilde{M}_j \quad \text{and} \quad K_{j+1}(P;\boldsymbol{\phi},0) \ll P^{1+\varepsilon}\tilde{H}_j\tilde{M}_j.$$

The first estimate follows as in the proof of Theorem 3.10. Also, on considering the implicit diophantine equations, we have

$$K_{j+1}(P;\boldsymbol{\phi},0) \ll PK_j^*(P;\boldsymbol{\phi}),$$

where $K_j^*(P;\boldsymbol{\phi})$ denotes the number of solutions of the system of equations (3.8), subject to our revised definition of J. Hence the same analysis as in the proof of Theorem 3.10 gives the desired conclusion.

Finally, if none of (i)-(iii) hold, we use (3.37), as in Theorem 3.10, in the above analysis to obtain estimate (b).

This completes the proof of the theorem.

4. Major and minor arc estimates

We must now obtain estimates of use in a Hardy-Littlewood dissection. Broadly speaking, we follow the pattern established by §3 of Vaughan [**8**]. As a consequence of the more efficient differencing procedure of Wooley [**13**], however, we have more variables to average over. We use an argument based on the large sieve to make some savings on these extra variables. Also, we develop particularly precise estimates for certain exponential sums, these enabling us to obtain an essentially best possible result for a $(k-j+1)$th power mean value estimate for F_j over the major arcs.

Throughout this section, we shall suppose that $1 \le j \le k-2$. When C is a non-zero integer, and $\mathcal{B} = \mathcal{B}(h_{j+1}, \dots, h_{k-2})$ is a subinterval of $[0, P_j]$, we define

$$D_j(\alpha; P, \boldsymbol{\phi}; \mathcal{B}, C)$$
$$= \sum_{h_1 \le 2^{j-1} H_1} \cdots \sum_{h_j \le H_j} \sum_{h_{j+1} \le P_j} \cdots \sum_{h_{k-2} \le P_j} \left| \sum_{z \in \mathcal{B}} e(C\alpha h_1 \dots h_{k-2} \xi^2) \right|^2,$$

where we write $\xi = 2z + h_{j+1} + \dots + h_{k-2}$. We then define

$$D_j(\alpha; P, \boldsymbol{\phi}) = \sup_{C \le \varepsilon^{-1}} \sup_{\mathcal{B}} D_j(\alpha; P, \boldsymbol{\phi}; \mathcal{B}, C). \tag{4.1}$$

Lemma 4.1. *Suppose that $(a,q) = 1$ and $|\alpha - a/q| \le q^{-2}$. Then*

$$D_j(\alpha; P, \boldsymbol{\phi}) \ll P^{\varepsilon} \left(\frac{Q_j^k}{q + Q_j^k |\alpha q - a|} + P^{-1} Q_j^k + q + Q_j^k |\alpha q - a| \right).$$

Proof. This is only a slight elaboration on the proof of Lemma 3.1 of Vaughan [**8**].

We shall suppose throughout Lemmata 4.2 to 4.6 that J, H, M are positive real numbers with $J \ll P^k$, $M \ll P^{1/k}$, and $H \ll PM^{-k}$. As a notational convenience, we shall also write Q^k for JH^3M^{2k}. When $\mathcal{C}$ is a subset of $\mathbb{Z} \cap (M, MR]$, we define

$$E_r(\alpha; J, H, M; \mathcal{C}) = \sum_{j \le J} \sum_{h \le H} \left| \sum_{m \in \mathcal{C}} e(\alpha j h^3 m^{2k}) \right|^{2r}. \tag{4.2}$$

Lemma 4.2. *Suppose that $M^k \le X \le Q^k M^{-k}$, and that $(a,q) = 1$, $q \le X$ and $|q\alpha - a| \le X^{-1}$. Then uniformly in $\mathcal{C}$, we have*

$$E_1(\alpha; J, H, M; \mathcal{C}) \ll P^{\varepsilon} \left(\frac{JHM^2}{(q + Q^k |\alpha q - a|)^{1/k}} + JHM + P^2 H \right).$$

Proof. We may apply the argument of the proof of Lemma 3.2 of Vaughan [**8**] to show that the sum in question is

$$\ll E + P^{\varepsilon} \left(JHM + H^3 M^{2k} \right),$$

where

$$E \ll P^{\varepsilon} \left(JHM + \frac{JHM^2}{(q + Q^k |\alpha q - a|)^{1/k}} \right).$$

This completes the proof of the lemma.

Corollary 4.2.1. *Suppose that* $k-j\geq 4$, $M_1^k\leq X\leq Q_j^kM_1^{-k}$, $(a,q)=1$, $q\leq X$ *and* $|q\alpha-a|\leq X^{-1}$. *Then uniformly in* $\mathcal{C}$, *we have*

$$\sup_{C\leq\varepsilon^{-1}} E_1(\alpha;CP^{k-j-2}H_1^{-1}\tilde{H}_j,H_1,M_1;\mathcal{C}) \ll P^{k-j-2+\varepsilon}\tilde{H}_jM_1^2\left((q+Q_j^k|\alpha q-a|)^{-1/k}+M_1^{-1}\right).$$

Proof. We merely note that when $k-j\geq 4$, we have $P^{k-j-2}\tilde{H}_j\gg P^2H_1$.

When $k-j\leq 3$ the following lemma usually provides a bound superior to that of Lemma 4.2.

Lemma 4.3. *Suppose that*

$$Y\leq\min\{M,JH^{-3},Q^{1/4},(Q^kM^{1-4k})^{1/6}\},\tag{4.3}$$

that $Y^k\leq X\leq Q^kY^{-k}$, *and that* $(a,q)=1$, $q\leq X$, *and* $|q\alpha-a|\leq X^{-1}$. *Then uniformly in* $\mathcal{C}$, *we have*

$$E_1(\alpha;J,H,M;\mathcal{C})\ll P^\varepsilon\left(\frac{JHM^2}{(q+Q^k|\alpha q-a|)^{1/k}}+JHM^2Y^{-1}\right).$$

Proof. The exponential sum in question is at most

$$\sum_{M<m_1,m_2\leq MR}\ \sum_{h\leq H}\min\{J,\|\alpha(m_2^{2k}-m_1^{2k})h^3\|^{-1}\}.$$

Since $Y\leq M$, the contribution from terms with $m_1=m_2$, combined with that from any terms with

$$\|\alpha(m_2^{2k}-m_1^{2k})h^3\|^{-1}\leq 4JY^{-1},$$

is $\ll P^\varepsilon HJM^2Y^{-1}$. Thus we need only consider

$$\sum_{m_1,m_2}\sum_{h\leq H}J\big(1+J\|\alpha(m_2^{2k}-m_1^{2k})h^3\|\big)^{-1},\tag{4.4}$$

where the first summation is over m_1 and m_2 satisfying

$$M<m_1<m_2\leq MR\quad\text{and}\quad\|\alpha(m_2^{2k}-m_1^{2k})h^3\|<(4J)^{-1}Y.\tag{4.5}$$

For given m_1,m_2,h, we may choose n so that

$$\|\alpha(m_2^{2k}-m_1^{2k})h^3\|=|\alpha(m_2^{2k}-m_1^{2k})h^3-n|.$$

Let $\mathcal{R} = (4JH^3Y^{-1})^{1/2}$. Then for given m_1, m_2, by Dirichlet's theorem we may choose b, r with

$$(b,r) = 1, \quad r \le \mathcal{R} \quad \text{and} \quad |\alpha(m_2^{2k} - m_1^{2k}) - b/r| \le (r\mathcal{R})^{-1}.$$

Notice that if $b = 0$ then $r = 1$. Hence, for any m_1, m_2, h included in the above sum we have

$$|bh^3 - nr| = |b/r - n/h^3|rh^3 < 2\left(\frac{YH^3}{4J}\right)^{1/2} \le 1,$$

since by assumption, $Y \le JH^{-3}$. Thus $bh^3 = nr$, and if $n = 0$ then $b = 0$ and $r = 1$. Hence in all cases $r|h^3$. Put $r = r_1r_2^2r_3^3$ where r_3 is maximal and $(r_1, r_2) = 1$. Then $r_1r_2r_3|h$. Let $h_0 = h/(r_1r_2r_3)$. Then the sum over h in (4.4) is

$$\begin{aligned}
&\ll \sum_{h_0 \le H/(r_1r_2r_3)} \frac{J}{1 + Jh_0^3(r_1r_2r_3)^3|\alpha(m_2^{2k} - m_1^{2k}) - b/r|} \\
&\ll \frac{JH(r_1r_2r_3)^{-1}}{(1 + JH^3|\alpha(m_2^{2k} - m_1^{2k}) - b/r|)^{1/3}} \\
&\ll \frac{JHr^{-1/3}}{(1 + JH^3|\alpha(m_2^{2k} - m_1^{2k}) - b/r|)^{1/3}}.
\end{aligned}$$

Thus

$$E_1(\alpha) \ll P^\varepsilon(A + JHM^2Y^{-1}),$$

where

$$A = \sum_{m_1, m_2} \frac{JH}{(r + JH^3|\alpha(m_2^{2k} - m_1^{2k})r - b|)^{1/3}},$$

and the summation is over m_1 and m_2 satisfying (4.5). Plainly, we may also restrict the summation to be with

$$r + JH^3|\alpha(m_2^{2k} - m_1^{2k})r - b| \le R^{-k}Y^3. \tag{4.6}$$

We put

$$j = (m_1, m_2), \quad n = m_1/j, \quad l = (m_2 - m_1)/j,$$

so that

$$j \le MR, \quad l \le MR/j, \quad M/j < n < n + l \le MR/j, \quad (n, n+l) = 1.$$

Now, of course, b and r will depend on j, l, n. Let $S = ((MR/j)^{2k-1}H^3J)^{1/2}$. Then given j and l, by Dirichlet's theorem we may choose c, s with

$$(c,s) = 1, \quad s \le S \quad \text{and} \quad |\alpha j^{2k}l - c/s| \le (sS)^{-1}.$$

Again we observe that if $c = 0$, then $s = 1$. Let $D = \big((n+l)^{2k} - n^{2k}\big)/l$. Then

$$D = \frac{2k}{l} \int_n^{n+l} x^{2k-1} dx,$$

and so

$$2k\left(\frac{M}{j}\right)^{2k-1} \le D \le 2k\left(\frac{MR}{j}\right)^{2k-1}.$$

Thus condition (4.6) implies that $r \le R^{-k} Y^3$, and

$$\left|\alpha j^{2k} l - \frac{b}{rD}\right| \le \frac{Y^3}{rDH^3JR^k}.$$

Therefore

$$\begin{aligned} |crD - bs| &= \left|\frac{c}{s} - \frac{b}{rD}\right| srD \\ &\le \frac{Y^3 D}{SR^k} + \frac{Y^3 S}{H^3JR^k} \\ &\le 4kR^{-k}Y^3(H^3J)^{-1/2}(MR/j)^{k-1/2} \\ &< 1, \end{aligned}$$

since by assumption, $Y^6 \le Q^k M^{1-4k} = H^3JM^{1-2k}$. Thus $crD = bs$. Hence $r|s$. Let $s_1 = s/r$. Then $cD = s_1 b$. Hence $c|b$ and $s_1|D$. Therefore, as $(n, n+l) = 1$, we have $(n(n+l), s_1) = 1$ and we may conclude that

$$A \ll \sum_{j \le MR} \sum_{l \le MR/j} \sum_{s_1|s} \sum_{n} \frac{HJ(s_1/s)^{1/3}}{(1 + H^3J(M/j)^{2k-1}|\alpha j^{2k} l - c/s|)^{1/3}}, \tag{4.7}$$

where the final summation is over n satisfying

$$n \le MR/j, \quad (n(n+l), s_1) = 1 \quad \text{and} \quad s_1|D. \tag{4.8}$$

By a simple argument, as in the proof of Lemma 3.2 of Vaughan [**8**] (see pages 22,23), there are $O((s_1 l)^\varepsilon)$ choices of n (mod s_1) satisfying (4.8). Thus the innermost sum in (4.7) is

$$\ll \left(\frac{MR}{js_1} + 1\right) \frac{P^\varepsilon (s_1/s)^{1/3} HJ}{(1 + H^3J(M/j)^{2k-1}|\alpha j^{2k} l - c/s|)^{1/3}}. \tag{4.9}$$

The contribution to A from terms in (4.7) with $MR \le js_1$ is therefore $\ll P^\varepsilon MHJ$. Thus, from (4.7) and (4.9), we have

$$A \ll P^\varepsilon (B + JHM^2Y^{-1}),$$

where

$$B=\sum_{j\le MR}\sum_{l\le MR/j}\frac{HJMj^{-1}}{(s+H^3J(M/j)^{2k-1}|\alpha j^{2k}ls-c|)^{1/3}}. \tag{4.10}$$

Plainly, we may restrict the second summation in (4.10) to those l satisfying

$$s+H^3J(M/j)^{2k-1}|\alpha j^{2k}ls-c|<R^{-2}(Y/j)^3.$$

Let $T=(M/j)^k(H^3J)^{1/2}$. Then given j, by Dirichlet's theorem we may choose d and t with

$$(d,t)=1,\quad t\le T\quad\text{and}\quad |\alpha j^{2k}-d/t|\le (tT)^{-1}.$$

Once again, if $d=0$ then $t=1$. Then for j and l included in the summation in (4.10), we have

$$\left|\frac{c}{ls}-\frac{d}{t}\right|lst\le\frac{2R^{-1}(Y/j)^3}{(H^3J)^{1/2}(M/j)^{k-1}}<1,$$

since, by assumption, $Y^6\le Q^kM^{1-4k}\le H^3J$. Thus $ct=dsl$, and so $s|t$. Let $t_1=t/s$. Then $ct_1=dl$. Thus $t_1|l$. Let $l_1=l/t_1$. Then $c=dl_1$. Therefore

$$B\le\sum_{j\le MR}\sum_{t_1|t}(t_1/t)^{1/3}\sum_{l_1\le MR/jt_1}\frac{HJMj^{-1}}{(1+H^3J(M/j)^{2k-1}l_1t_1|\alpha j^{2k}-d/t|)^{1/3}}.$$

By Lemma 7.1 of Vaughan and Wooley [**10**], the innermost sum is

$$\ll\frac{HJM^2Rj^{-2}t_1^{-1}}{(1+H^3J(M/j)^{2k}|\alpha j^{2k}-d/t|)^{1/3}}.$$

Thus

$$B\ll P^{\varepsilon}(C+JHM^2Y^{-1})$$

where

$$C=\sum_{j\le MR}\frac{HJM^2j^{-2}}{(t+H^3J(M/j)^{2k}|\alpha j^{2k}t-d|)^{1/3}},$$

and we may restrict the summation to those j satisfying

$$t+H^3J(M/j)^{2k}|\alpha j^{2k}t-d|<\tfrac12(Y/j)^3. \tag{4.11}$$

Let $U=Q^{k/2}$. Then by Dirichlet's theorem we may choose e,u with

$$(e,u)=1,\quad u\le U\quad\text{and}\quad |\alpha u-e|\le U^{-1}.$$

On noting that t is non-zero, we find that for any j satisfying (4.11), we have $j\le Y$. Then when j satisfies (4.11), we have

$$\begin{aligned}\left|\frac{e}{u}-\frac{d}{j^{2k}t}\right|j^{2k}tu&<\tfrac12(Y/j)^3\left(\frac{j^{2k}}{U}+\frac{U}{H^3J(M/j)^{2k}}\right)\\&\le\frac{Y^{2k}}{2U}+\frac{Y^{2k}U}{2H^3JM^{2k}}\\&\le 1,\end{aligned}$$

since by assumption, $Y^{2k} \le Q^{k/2}$. Thus $ej^{2k}t = du$. Hence $t|u$. Let $u_0 = u/t$. Then $ej^{2k} = du_0$. Hence $u_0|j^{2k}$. Let $u_0 = u_1u_2^2 \dots u_{2k}^{2k}$ where u_{2k} is maximal and $u_1, \dots, u_{2k-1}$ are squarefree and coprime in pairs. Then $u_1 \dots u_{2k}|j$, whence

$$C \ll \sum_{w} \sum_{t,u_1,\dots,u_{2k}} \frac{HJM^2(wu_1u_2\dots u_{2k})^{-2}t^{-1/3}}{(1+H^3JM^{2k}|\alpha - e/u|)^{1/3}},$$

where the second summation is over $t, u_1, \dots, u_{2k}$ satisfying $tu_1u_2^2 \dots u_{2k}^{2k} = u$. Thus

$$C \ll \frac{P^{\varepsilon}HJM^2}{(u + Q^k|\alpha u - e|)^{1/k}}.$$

When $u + Q^k|\alpha u - e| \ge \frac{1}{2}Y^k$ we are done, so we may suppose that $u + Q^k|\alpha u - e| < \frac{1}{2}Y^k$. Thus

$$\left|\frac{e}{u} - \frac{a}{q}\right| uq < \frac{Y^kX}{2Q^k} + \frac{Y^k}{2X} \le 1,$$

since by assumption, $Y^k \le X \le Q^kY^{-k}$. Hence $eq = au$, so that $u = q$, $e = a$, and the bound for $E_1(\alpha)$ follows at once.

This completes the proof of the lemma.

In the next two lemmata we prepare a large sieve argument which yields a further useful bound on $E_s(\alpha)$. In Lemmata 4.4 to 4.6, the variable N denotes a large positive integer with $M^{2k} \ll N \ll P^k$. Then in particular, $JHN \gg Q^k$.

Lemma 4.4. *Let $c(n)$ $(n \in \mathbb{N})$ be arbitrary complex numbers, and define*

$$S(\beta) = \sum_{j=1}^{J} \left| \sum_{n=1}^{N} c(n)e(\beta jn) \right|^2.$$

Suppose that $(a,q) = 1$ and $|\beta - a/q| \le q^{-2}$. Then

$$S(\beta) \ll P^{\varepsilon}\left(\frac{JN}{q + JN|\beta q - a|} + J + N + q + JN|\beta q - a|\right) \sum_{n=1}^{N} |c(n)|^2.$$

Proof. On squaring out, interchanging the order of summation, and performing the summation over j, we find that

$$S(\beta) \ll J\sum_{n=1}^{N} |c(n)|^2 + \sum_{1 \le n_1 < n_2 \le N} |c(n_1)c(n_2)| \min\{J, \|\beta(n_2 - n_1)\|^{-1}\}. \quad (4.12)$$

Thus it suffices to treat the second term on the right hand side of (4.12), which by the arithmetic-geometric mean inequality is

$$\ll \sum_{1 \le n_1 < n_2 \le N} \left(|c(n_1)|^2 + |c(n_2)|^2\right) \min\{J, \|\beta(n_2 - n_1)\|^{-1}\}$$
$$\ll \sum_{n=1}^{N} |c(n)|^2 \sum_{h=1}^{N} \min\{J, \|\beta h\|^{-1}\}.$$

When $q \geq NJ$ the lemma follows trivially by Cauchy's inequality. Then we may suppose that $q < NJ$, and so by Lemma 2.2 of Vaughan [**5**] we have

$$S(\beta) \ll P^{\varepsilon}(NJq^{-1} + J + N + q) \sum_{n=1}^{N} |c(n)|^2. \tag{4.13}$$

If $NJ|\beta q - a| \leq q$ then we are done. We therefore suppose that $NJ|\beta q - a| > q$, and that a and q satisfy the hypotheses of the lemma.

By Dirichlet's theorem we may choose b and r with

$$(b, r) = 1, \quad r \leq 2|\beta q - a|^{-1} \quad \text{and} \quad |\beta r - b| \leq \tfrac{1}{2}|\beta q - a|.$$

It follows that $b/r \neq a/q$ and $|\beta r - b| \leq (2q)^{-1}$. Thus

$$(qr)^{-1} \leq |\beta - a/q| + |\beta - b/r| \leq |\beta - a/q| + (2qr)^{-1},$$

whence $(2|\beta q - a|)^{-1} \leq r$. Therefore, by (4.13) with q replaced by r, we have

$$\begin{aligned} S(\beta) &\ll P^{\varepsilon}\left(NJr^{-1} + J + N + r\right) \sum_{n=1}^{N} |c(n)|^2 \\ &\ll P^{\varepsilon}\left(JN|\beta q - a| + J + N + |\beta q - a|^{-1}\right) \sum_{n=1}^{N} |c(n)|^2, \end{aligned}$$

and the desired conclusion follows.

Lemma 4.5. *Let $c(n)$ $(n \in \mathbb{N})$ be arbitrary complex numbers, and define*

$$T(\alpha) = \sum_{j \leq J} \sum_{h \leq H} \left| \sum_{n=1}^{N} c(n) e(\alpha h^3 jn) \right|^2.$$

Suppose that

$$Y \leq \min\{N, J, (JNH^{-3})^{1/2}\},$$

that $Y^3 \leq X \leq Q^k Y^{-3}$, and that $(a, q) = 1$, $q \leq X$ and $|q\alpha - a| \leq X^{-1}$. Then

$$T(\alpha) \ll P^{\varepsilon}\left(\frac{JHN}{(q + Q^k|\alpha q - a|)^{1/3}} + JHNY^{-1}\right) \sum_{n=1}^{N} |c(n)|^2.$$

Proof. Let $S = NJY^{-1}$. Then given h, by Dirichlet's theorem we may choose c and s with

$$(c, s) = 1, \quad s \leq S \quad \text{and} \quad |\alpha h^3 s - c| \leq S^{-1}. \tag{4.14}$$

Thus, by Lemma 4.4 we have

$$T(\alpha) \ll P^{\varepsilon} \sum_{h \le H} \left(\frac{JN}{s + JN|\alpha h^3 s - c|} + J + N + s + JN|\alpha h^3 s - c| \right) \sum_{n=1}^{N} |c(n)|^2.$$

But by (4.14), we have

$$s + JN|\alpha h^3 s - c| \ll Y + NJY^{-1},$$

so in view of the hypotheses on the size of Y, we have

$$T(\alpha) \ll P^{\varepsilon} \left(\sum_{h \le H} \frac{JN}{s + JN|\alpha h^3 s - c|} + HJNY^{-1} \right) \sum_{n=1}^{N} |c(n)|^2.$$

Thus it remains to estimate

$$\sum_{h \le H} (s + JN|\alpha h^3 s - c|)^{-1}, \tag{4.15}$$

where, plainly, we may restrict the summation to those h with

$$s + JN|\alpha h^3 s - c| < \tfrac{1}{2} Y. \qquad (4.16).$$

Let $T = (H^3 JN)^{1/2}$. Then by Dirichlet's theorem we may choose d and t with

$$(d, t) = 1, \quad t \le T \quad \text{and} \quad |\alpha - d/t| \le (tT)^{-1}.$$

Then for each h satisfying (4.16), we have

$$|dh^3 s - ct| = \left| \frac{d}{t} - \frac{c}{h^3 s} \right| th^3 s < \frac{H^3 Y}{2T} + \frac{TY}{2JN} \le Y \left(\frac{H^3}{JN} \right)^{1/2} \le 1,$$

since by assumption, $Y^2 \le JNH^{-3}$. Thus $dh^3 s = ct$, and so $s|t$. Let $t_0 = t/s$. Then $dh^3 = ct_0$. Therefore, $t_0 | h^3$, so by puting $t_0 = t_1 t_2^2 t_3^3$ with t_3 maximal and t_1, t_2 squarefree, we have $t_1 t_2 t_3 | h$. Hence the sum (4.15) is

$$\begin{aligned}
&\ll \sum_{t_0 | t} \sum_{j \le H/(t_1 t_2 t_3)} \frac{(t_0/t)}{1 + JN(j t_1 t_2 t_3)^3 |\alpha - d/t|} \\
&\ll \sum_{t_0 | t} \frac{(t_0/t) H}{(t_1 t_2 t_3) \left(1 + JNH^3 |\alpha - d/t|\right)^{1/3}} \\
&\ll \frac{H t^{\varepsilon}}{(t + Q^k |\alpha t - d|)^{1/3}}.
\end{aligned}$$

If $t + Q^k|\alpha t - d| \ge \frac{1}{2} Y^3$, then we are done. Thus we may suppose that $t + Q^k|\alpha t - d| < \frac{1}{2} Y^3$. Therefore

$$|dq - at| = \left| \frac{d}{t} - \frac{a}{q} \right| tq < \frac{Y^3}{2X} + \frac{Y^3 X}{2Q^k} \le 1,$$

since by assumption, $Y^3 \le X \le Q^k Y^{-3}$. Hence $at = dq$, so that $q = t$ and $a = d$, and the bound for $T(\alpha)$ follows at once.

This completes the proof of the lemma.

Lemma 4.6. *Suppose that*

$$Y \le \min\{M^{2k}, J, (JM^{2k}H^{-3})^{1/2}\}, \tag{4.17}$$

that $Y^3 \le X \le Q^k Y^{-3}$*, and that* $(a,q)=1$*,* $q \le X$*, and* $|q\alpha - a| \le X^{-1}$*. Then uniformly in* $\mathcal{C}$ *satisfying* $\mathcal{C} \subseteq \mathcal{A}(MR,R) \cap (M, MR]$*, we have*

$$E_s(\alpha; J, H, M; \mathcal{C}) \ll P^\varepsilon JHM^{\mu_s+2k}\left(Y^{-1} + \left(q + Q^k|\alpha q - a|\right)^{-1/3}\right).$$

Proof. For $n \in \mathbb{N}$, define $c(n)$ to be the number of solutions of the diophantine equation

$$x_1^{2k} + \cdots + x_s^{2k} = n,$$

with $x_i \in \mathcal{C}$ $(1 \le i \le s)$. Also, let $N = (MR)^{2k}$. Then by (4.2), it follows that $E_s(\alpha; J, H, M; \mathcal{C})$ is an exponential sum of the form $T(\alpha)$ of Lemma 4.5. The lemma then follows on noting that $M^{2k} \ll N \ll P^k$, and

$$\sum_{n=1}^{N} |c(n)|^2 \ll S_s^{(2k)}(MR, R) \ll P^\varepsilon M^{\mu_s}.$$

We now attend to the matter of obtaining suitable major arc estimates for the exponential sums $F_j(\alpha)$.

Lemma 4.7. *Suppose that* $(a,q)=1$*,* $\beta = \alpha - a/q$*, and* $qP^{-1}Q_j^k R^{k(k-j)}|\beta| \le 1$*. Then*

$$F_j(\alpha) \ll \sum_{\mathbf{m}}\sum_{\mathbf{h}} \frac{Pq^{-1}\tau_j(q,a,\mathbf{h},\mathbf{m})}{(1+|\beta|h_1\dots h_j P^{k-j})^{\frac{1}{k-j}}} + \tilde{H}_j\tilde{M}_j q^{\frac{k-j-1}{k-j}+\varepsilon},$$

where the summation is over $\mathbf{h}$ *and* $\mathbf{m}$ *satisfying* (2.1)*, and*

$$\tau_j(q,a,\mathbf{h},\mathbf{m}) = \left|\sum_{r=1}^{q} e\left(\frac{a}{q}\Psi_j(r,\mathbf{h},\mathbf{m})\right)\right|. \tag{4.18}$$

Proof. The proof we give is a simple modification of the proof of Lemma 3.5 of Vaughan [**8**]. We have

$$F_j(\alpha) = \sum_{\mathbf{h}}\sum_{\mathbf{m}} S(\alpha; \mathbf{h}; \mathbf{m}), \tag{4.19}$$

where

$$S(\alpha; \mathbf{h}; \mathbf{m}) = \sum_{1 \le z \le P_j} e\left(\alpha\Psi_j(z; \mathbf{h}; \mathbf{m})\right).$$

Hence, on writing $\alpha = a/q + \beta$, a standard argument gives

$$S(\alpha; \mathbf{h}, \mathbf{m}) = q^{-1} \sum_{-\frac{1}{2}q < b \le \frac{1}{2}q} \sigma(q,a,b,\mathbf{h},\mathbf{m})T(\beta, b, \mathbf{h}, \mathbf{m}), \tag{4.20}$$

where

$$\sigma(q,a,b,\mathbf{h},\mathbf{m}) = \sum_{r=1}^{q} e\left(\frac{a}{q}\Psi_j(r,\mathbf{h},\mathbf{m}) + \frac{b}{q}r\right)$$

and

$$T(\beta,b,\mathbf{h},\mathbf{m}) = \sum_{1\le z\le P_j} e\left(\beta\Psi_j(z,\mathbf{h},\mathbf{m}) - \frac{b}{q}z\right).$$

Each coefficient of Ψ_j is divisible by $h_1 \dots h_j$, and so if d is the greatest common divisor of the coefficients of $a\Psi_j(r,\mathbf{h},\mathbf{m}) + br$ and q, then $d \ll (q, h_1 \dots h_j, b)$. Therefore by Theorem 7.1 of Vaughan [**5**], we have

$$\sigma(q,a,b,\mathbf{h},\mathbf{m}) \ll q^{\frac{k-j-1}{k-j}+\varepsilon}(q,h_1\dots h_j,b)^{\frac{1}{k-j}}. \tag{4.21}$$

Let

$$\phi(\gamma) = \beta\Psi_j(\gamma,\mathbf{h},\mathbf{m}) - \frac{b}{q}\gamma. \tag{4.22}$$

Then

$$\frac{b}{q} + \phi'(\gamma) = \frac{k!}{(k-j-1)!}\beta(m_1\dots m_j)^{-k}I,$$

where

$$I = \int_{\gamma-h_1m_1^k}^{\gamma+h_1m_1^k}\int_{\psi_1-h_2m_2^k}^{\psi_1+h_2m_2^k}\cdots\int_{\psi_{j-1}-h_jm_j^k}^{\psi_{j-1}+h_jm_j^k}\psi_j^{k-j-1}d\psi_j d\psi_{j-1}\dots d\psi_1.$$

Thus, when $|\gamma| \le 2^jP$, we have

$$\begin{aligned}\left|\frac{b}{q} + \phi'(\gamma)\right| &\le 2^j\frac{k!}{(k-j-1)!}|\beta|h_1\dots h_j(2^jP + h_1m_1^k + \dots + h_jm_j^k)^{k-j-1} \\ &< \frac{1}{4q}.\end{aligned}$$

When $-\frac{1}{2}q < b \le \frac{1}{2}q$ and $|\gamma| \le 2^jP$, we therefore have $|\phi'(\gamma)| < \frac{3}{4}$. Further, when $b \ne 0$ we have

$$|\phi'(\gamma)| > \frac{|b|}{2q}.$$

Therefore, by Lemma 4.2 of Vaughan [**5**], we have

$$T(\beta,b,\mathbf{h},\mathbf{m}) = \sum_{u=-1}^{1} I(\beta,b,\mathbf{h},\mathbf{m},u) + O(1),$$

where

$$I(\beta,b,\mathbf{h},\mathbf{m},u) = \int_0^{2^jP} e(\phi(\gamma)-\gamma u)d\gamma. \tag{4.23}$$

By integrating by parts we deduce that

$$I(\beta, b, \mathbf{h}, \mathbf{m}, \pm 1) \ll 1,$$

and further, when $b \neq 0$,

$$I(\beta, b, \mathbf{h}, \mathbf{m}, 0) \ll \frac{q}{|b|}.$$

Therefore

$$T(\beta, 0, \mathbf{h}, \mathbf{m}) = I(\beta, 0, \mathbf{h}, \mathbf{m}, 0) + O(1),$$

and, when $b \neq 0$,

$$T(\beta, b, \mathbf{h}, \mathbf{m}) \ll \frac{q}{|b|}.$$

Hence, by (4.20) and (4.21), we have

$$\begin{aligned} S(\alpha, \mathbf{h}, \mathbf{m}) - q^{-1}\sigma(q, a, 0, \mathbf{h}, \mathbf{m}) I(\beta, 0, \mathbf{h}, \mathbf{m}, 0) &\ll \sum_{1 \le b \le \frac{1}{2}q} b^{-1} q^{\frac{k-j-1}{k-j}+\varepsilon} (q, b)^{\frac{1}{k-j}} \\ &\ll q^{\frac{k-j-1}{k-j}+\varepsilon}. \end{aligned}$$

The lemma now follows from (4.19) on observing that by (4.22), (4.23) and Theorem 7.3 of Vaughan [**5**], we have

$$I(\beta, 0, \mathbf{h}, \mathbf{m}, 0) \ll P(1 + |\beta| h_1 \ldots h_j P^{k-j})^{-\frac{1}{k-j}}.$$

In the following lemma we provide an estimate for an exponential sum which we will use ultimately to estimate $\tau_j(q, a, \mathbf{h}, \mathbf{m})$ when $j \le k-3$.

Lemma 4.8. *Suppose that* $n \ge 2$. *When* $q \in \mathbb{N}$ *and* $a_1, \ldots, a_n \in \mathbb{Z}$, *define* $f(x; \mathbf{a}) = \sum_{j=1}^{n} a_j x^j$ *and*

$$S(q, \mathbf{a}) = \sum_{x=1}^{q} e\left(\frac{f(x; \mathbf{a})}{q}\right).$$

Let $d = (q, a_1, \ldots, a_n)$ *and* $r = q/d$. *Define* r_j $(1 \le j \le n)$ *by*

$$r_j = \prod_{p^j \| r} p^j \quad (1 \le j < n), \quad \text{and} \quad r_n = \prod_{p^i \| r,\ i \ge n} p^i.$$

Then

$$S(q, \mathbf{a}) \ll q^{\varepsilon} d r_1^{1/2} \prod_{j=2}^{n} r_j^{1-1/j}.$$

Proof. Let $r = q/d$, and $b_j = a_j/d$ $(1 \le j \le n)$. Then $S(q, \mathbf{a}) = dS(r, \mathbf{b})$, with $(r, b_1, \ldots, b_n) = 1$. In view of the multiplicative property of $S(q, \mathbf{b})$ (see the proof of Theorem 7.1 of Vaughan [**5**]), it suffices to treat the case in which r is a prime power, say p^t. Suppose that $(p, c_1, \ldots, c_n) = 1$. Then by Corollary 2F of Chapter II of Schmidt [**3**], we have

$$S(p, \mathbf{c}) \ll p^{1/2},$$

and by Theorem 7.1 of Vaughan [**5**], for each $t \geq n$ we have

$$S(p^t, \mathbf{c}) \ll p^{t-t/n}.$$

Thus we may assume that $2 \leq t \leq n-1$.

By making the transformation $x \longmapsto u + vp^{t-1}$ with $1 \leq u \leq p^{t-1}$, $1 \leq v \leq p$, we have

$$\begin{aligned} S(p^t, \mathbf{c}) &= \sum_{u=1}^{p^{t-1}} \sum_{v=1}^{p} e\left(\frac{f(u; \mathbf{c})}{p^t} + \frac{f'(u; \mathbf{c})v}{p}\right) \\ &= \sum_{u=1}^{p^{t-1}} pe\left(\frac{f(u; \mathbf{c})}{p^t}\right), \end{aligned}$$

where the final summation includes only those u with $p | f'(u; \mathbf{c})$. But since

$$(p, c_1, 2c_2, \ldots, nc_n) \leq n(p, c_1, \ldots, c_n) = n,$$

the congruence $f'(u; \mathbf{c}) \equiv 0 \pmod p$ has at most $n(n-1)$ solutions $\pmod p$, say $\xi_1, \ldots, \xi_N$. Thus

$$S(p^t, \mathbf{c}) = \sum_{w=1}^{p^{t-2}} \sum_{j=1}^{N} pe\left(\frac{f(\xi_j + wp; \mathbf{c})}{p^t}\right) \ll n(n-1)p^{t-1},$$

and this completes the proof of the lemma.

We are now able to establish a suitable estimate for a moment of $F_j(\alpha)$ of use on the major arcs.

Definition 4.9.

(i) Let $\mathfrak{m}_j$ denote the set of points in $[0, 1]$ with the property that whenever there are $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a, q) = 1$, and

$$qP^{-1}Q_j^k R^{k(k-j)} |\alpha - a/q| \leq 1, \tag{4.24}$$

then $q > P$. Further, let $\mathfrak{M}_j = [0, 1] \setminus \mathfrak{m}_j$.

(ii) When $(a, q) = 1$, let $\mathfrak{M}_j(q, a)$ be the set of α in $[0, 1]$ for which (4.24) holds. (Note that the $\mathfrak{M}_j(q, a)$ with $0 \leq a \leq q \leq P$ are disjoint.)

(iii) Define $F_j^*(\alpha)$ to be the function of α taking the value zero whenever $\alpha \in \mathfrak{m}_j$, and by

$$F_j^*(\alpha) = \sum_{\mathbf{m}} \sum_{\mathbf{h}} \frac{Pq^{-1}\tau_j(q, a, \mathbf{h}, \mathbf{m})}{(1 + |\beta| h_1 \ldots h_j P^{k-j})^{\frac{1}{k-j}}}$$

whenever $\alpha \in \mathfrak{M}_j(q, a)$ and $0 \leq a \leq q \leq P$. Here τ_j is defined as in (4.18), and we have written β for $\alpha - a/q$.

Lemma 4.10. *Suppose that* $1 \le j \le k-3$ *and* $t \ge k-j+1$. *Then*

$$\int_0^1 |F_j^*(\alpha)|^t d\alpha \ll P^\varepsilon (P\tilde{H}_j\tilde{M}_j)^t Q_j^{-k}.$$

Proof. The integral to be estimated is

$$\sum_{q\le P}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{\mathfrak{M}_j(q,a)}\left(\sum_{\mathbf{m}}\sum_{\mathbf{h}}\frac{Pq^{-1}\tau_j(q,a,\mathbf{h},\mathbf{m})}{(1+|\alpha-a/q|h_1\dots h_jP^{k-j})^{\frac{1}{k-j}}}\right)^t d\alpha.$$

Let $h = h_1\dots h_j$, and for a typical q from the summation, put $r = q/(q,h)$. Write $r=\prod_{i=1}^{k-j} r_i$, where the r_i $(1\le i\le k-j)$ are defined as in the statement of Lemma 4.8. On recalling the definition of Ψ_j, and applying Lemma 4.8 to (4.18), we obtain

$$\tau_j(q,a,\mathbf{h},\mathbf{m}) \ll q^\varepsilon (q,h) r_1^{1/2}\prod_{i=2}^{k-j} r_i^{1-1/i}.$$

Hence

$$\sum_{\mathbf{m}}\sum_{\mathbf{h}}\frac{Pq^{-1}\tau_j(q,a,\mathbf{h},\mathbf{m})}{(1+|\alpha-a/q|h_1\dots h_jP^{k-j})^{\frac{1}{k-j}}} \ll P^{1+\varepsilon}\tilde{M}_j J(q,\tilde{H}_j), \tag{4.25}$$

where

$$J(q,H)=\sum_{h\le H}\frac{r_1^{-1/2}\prod_{i=2}^{k-j} r_i^{-1/i}}{(1+|\alpha-a/q|hP^{k-j})^{\frac{1}{k-j}}}.$$

Here, of course, the r_i depend implicitly on both q and h. We may classify the values of h in the last summation according to the size of $d=(q,h)$. Thus we deduce that

$$J(q,H)\ll \sum_{dr=q} r_1^{-1/2}\prod_{i=2}^{k-j} r_i^{-1/i}\frac{Hd^{-1}}{(1+|\alpha-a/q|HP^{k-j})^{\frac{1}{k-j}}}, \tag{4.26}$$

where $r=\prod_{i=1}^{k-j} r_i$, as in the statement of Lemma 4.8. Therefore, by (4.25), (4.26) and Hölder's inequality, we obtain

$$\sum_{q\le P}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{\mathfrak{M}_j(q,a)}|F_j^*(\alpha)|^t d\alpha \ll P^\varepsilon\left(P\tilde{H}_j\tilde{M}_j\right)^t J\sigma(q), \tag{4.27}$$

where

$$J=\int_0^1\frac{d\beta}{\left(1+\beta Q_j^k\right)^{\frac{t}{k-j}}},$$

and

$$\sigma(q) = \sum_{q \le P} q \sum_{dr=q} \left(d^{-1} r_1^{-1/2} \prod_{i=2}^{k-j} r_i^{-1/i} \right)^t .$$

We have

$$J \ll Q_j^{-k}. \tag{4.28}$$

Also, on noting that

$$\begin{aligned}\sigma(q) &\le \sum_{d \le P} d^{1-t} \sum_{r \le P} r \left(r_1^{-1/2} \prod_{i=2}^{k-j} r_i^{-1/i} \right)^t \\ &\ll \prod_{p \le P} \left(1 + p^{1-t/2} + \sum_{i=2}^{k-j} p^{i-t} + \sum_{i=k-j+1}^{\infty} p^{i - \frac{ti}{k-j}} \right),\end{aligned}$$

we deduce that for some fixed κ, we have

$$\sigma(q) \ll \prod_{p \le P} \left(1 + \kappa p^{-1} \right) \ll P^{\varepsilon}. \tag{4.29}$$

The lemma now follows on combining (4.27), (4.28) and (4.29).

5. The iterative scheme for fifth powers, I

The iterative scheme for $s > 6$ is rather more complicated than that for $s \le 6$. We defer the treatment of the former cases to §6. For $s = 1$ and 2 we have the classical bounds

$$S_s(P, R) \ll P^{s+\varepsilon},$$

and for $s = 3$ and 4 we use the results of Theorem 1.4 of Vaughan [**9**]. These give

$$\lambda_3^* = 3 + 2\theta \quad \text{and} \quad \lambda_4^* = \frac{4 + 2\theta}{1 - \theta},$$

where θ is the smallest non-negative root of the polynomial $3 - 42\theta - 27\theta^2 - 42\theta^3$. Thus we obtain $\lambda_3^* \le 3.136258$ and $\lambda_4^* \le 4.438657$. We display below the iterative procedures we adopt for $s = 5$ and 6.

$s = 5$.

$$\begin{array}{ccccccc} F_0^2 f_0^8 & \longmapsto & F_1 f_1^8 & \longrightarrow & F_2 f_2^6 & \Longrightarrow & (F_2^4)^{1/4} (f_2^8)^{3/4} \\ & & & & \big\downarrow & & \\ & & & & f_1^{10} & & \end{array}$$

$s = 6$.

$$\begin{array}{ccccccc} F_0^2 f_0^{10} & \longmapsto & F_1 f_1^{10} & \longrightarrow & F_2 f_2^8 & \Longrightarrow & (F_2^4)^{1/4} (f_2^{10})^{1/2} (f_2^{12})^{1/4} \\ & & & & \big\downarrow & & \\ & & & & f_1^{12} & & \end{array}$$

In what follows, we let (λ_s) be an iterate of the sequence converging to (λ_s^*), and we write θ for ϕ_1 and ϕ for ϕ_2 . Note that to obtain a reasonable initial iterate (λ_s), we may use the values given by Lemma 3.2 of Wooley [**13**].

(i) $s = 5$.

By Theorem 3.11(I) case (i), we have

$$\int_0^1 |F_2(\alpha)|^4 d\alpha \ll P^{2+\varepsilon} \tilde{H}_2^3 \tilde{M}_2^3. \tag{5.1}$$

Then proceeding as described in §2, using the iterative sequence for $s = 5$ given above, the equations for λ_5, θ, and ϕ are determined by

$$PH_1M_1M_2Q_2^{\lambda_3^*} \approx \big(P^2(H_1H_2M_1M_2)^3\big)^{1/4}\big(Q_2^{\lambda_4^*}\big)^{3/4}, \tag{5.2}$$

$$PM_1Q_1^{\lambda_4^*} \approx \big(P(M_1H_1)^2M_2^6Q_2^{\lambda_3^*}Q_1^{\lambda_5}\big)^{1/2}, \tag{5.3}$$

$$P^{\lambda_5} \approx PM_1^8Q_1^{\lambda_4^*}. \tag{5.4}$$

On writing

$$\delta = \tfrac{3}{4}\lambda_4^* - \lambda_3^*, \tag{5.5}$$

equation (5.2) leads to the equation

$$\delta(1-\theta-\phi) + \theta - 4\phi = 0,$$

and hence

$$\phi = \frac{\theta + \delta(1-\theta)}{4+\delta}. \tag{5.6}$$

Meanwhile, equation (5.3) leads to the equation

$$2(1+\theta+\lambda_4^*(1-\theta)) = \lambda_5(1-\theta) + \lambda_3^*(1-\theta-\phi) + 3 - 8\theta + 6\phi.$$

On writing $\mathcal{E} = \lambda_5 - 2\lambda_4^* + \lambda_3^*$, we obtain

$$\mathcal{E}(1-\theta) + 1 - 10\theta = (\lambda_3^* - 6)\phi. \tag{5.7}$$

Write

$$\alpha = \frac{\lambda_3^* - 6}{4+\delta}. \tag{5.8}$$

Then (5.6) and (5.7) yield

$$\theta = \frac{1 + \mathcal{E} - \alpha\delta}{10 + \mathcal{E} + \alpha(1-\delta)}.$$

By (5.4), the next iterate for λ_5 is therefore given by

$$\lambda_5' = \lambda_4^*(1-\theta) + 1 + 8\theta.$$

The above iteration process converges to λ_5^*, with

$$\lambda_5^* = \lambda_4^*(1-\theta_5) + 1 + 8\theta_5, \tag{5.9}$$

where θ_5 is a root of the equation given by substituting the expression (5.9) into

$$(10 + \mathcal{E}^* + \alpha(1-\delta))\theta_5 = 1 + \mathcal{E}^* - \alpha\delta,$$

with $\mathcal{E}^* = \lambda_5^* - 2\lambda_4^* + \lambda_3^*$. We find that

$$\mathcal{E}^* = \lambda_3^* + 1 + 8\theta_5 - \lambda_4^*(1+\theta_5),$$

and so

$$(8-\lambda_4^*)\theta_5^2 + (3 + \lambda_3^* + \alpha(1-\delta))\theta_5 - (2 + \lambda_3^* - \lambda_4^* - \alpha\delta) = 0, \tag{5.10}$$

with δ and α given by (5.5) and (5.8) respectively. It transpires that θ_5 is the positive root of equation (5.10), whence $\lambda_5^* \le 5.925080$.

(ii) $s = 6$.

We observe that the estimate (5.1) holds once again. Then proceeding as described in §2, using the iterative sequence for $s = 6$ given above, the equations for λ_6, θ, and ϕ are determined by

$$PH_1M_1M_2Q_2^{\lambda_4^*} \approx \left(P^2(H_1H_2M_1M_2)^3\right)^{1/4}\left(Q_2^{\lambda_5^*}\right)^{1/2}\left(Q_2^{\lambda_6}\right)^{1/4}, \tag{5.11}$$

$$PM_1Q_1^{\lambda_5^*} \approx \left(P(M_1H_1)^2M_2^8Q_2^{\lambda_4^*}Q_1^{\lambda_6}\right)^{1/2}, \tag{5.12}$$

$$P^{\lambda_6} \approx PM_1^{10}Q_1^{\lambda_5^*}. \tag{5.13}$$

On writing $\delta' = \frac{1}{2}\lambda_5^* + \frac{1}{4}\lambda_6 - \lambda_4^*$, equation (5.11) leads to the equation

$$\delta'(1-\theta-\phi) + \theta - 4\phi = 0,$$

and hence

$$\phi = \frac{\theta + \delta'(1-\theta)}{4+\delta'}. \tag{5.14}$$

Meanwhile, equation (5.12) leads to the equation

$$2(1 + \theta + \lambda_5^*(1-\theta)) = \lambda_6(1-\theta) + \lambda_4^*(1-\theta-\phi) + 3 - 8\theta + 8\phi.$$

On writing $\mathcal{E}' = \lambda_6 - 2\lambda_5^* + \lambda_4^*$, we obtain

$$\mathcal{E}'(1-\theta) + 1 - 10\theta = (\lambda_4^* - 8)\phi. \tag{5.15}$$

Write
$$\alpha' = \frac{\lambda_4^* - 8}{4 + \delta'}.$$
Then (5.14) and (5.15) yield
$$\theta = \frac{1 + \mathcal{E}' - \alpha'\delta'}{10 + \mathcal{E}' + \alpha'(1 - \delta')}.$$
By (5.13), the next iterate for λ_6 is therefore given by
$$\lambda_6' = \lambda_5^*(1 - \theta) + 1 + 10\theta.$$

The above iteration process converges to λ_6^*, with
$$\lambda_6^* = \lambda_5^*(1 - \theta_6) + 1 + 10\theta_6, \tag{5.16}$$
where θ_6 is a root of the equation
$$(10 + \mathcal{E}^* + \alpha^*(1 - \delta^*))\theta_6 = 1 + \mathcal{E}^* - \alpha^*\delta^*,$$
in which
$$\delta^* = \tfrac{1}{2}\lambda_5^* + \tfrac{1}{4}\lambda_6^* - \lambda_4^*, \quad \mathcal{E}^* = \lambda_6^* - 2\lambda_5^* + \lambda_4^*, \quad \alpha^* = \frac{\lambda_4^* - 8}{4 + \delta^*},$$
and in δ^*, $\mathcal{E}^*$ and α^* we substitute for λ_6^* from (5.16).

The root of the resulting cubic polynomial can be found directly. Alternatively, one may continue the iteration process to obtain a good approximation to the root. Thus, by (5.16) we obtain $\lambda_6^* \le 7.541755$.

6. The iterative scheme for fifth powers, II

We display below the iterative procedures we adopt for $s = 7$ and 8.

$s = 7$.
$$\begin{array}{ccccccc} F_0^2 f_0^{12} & \longmapsto & F_1 f_1^{12} & \longrightarrow & F_2 f_2^{12} & \Longrightarrow & (F_2)(f_2^{12}) \\ & & & & \downarrow & & \\ & & & & f_1^{12} & & \end{array}$$

$s = 8$.
$$\begin{array}{ccccccc} F_0^2 f_0^{14} & \longmapsto & F_1 f_1^{14} & \longrightarrow & F_2 f_2^{12} & \Longrightarrow & (F_2)(f_2^{12}) \\ & & & & \downarrow & & \\ & & & & f_1^{16} & & \end{array}$$

The iterative procedures for λ_7 and λ_8 must be taken together. Before we go on to explain the iterative procedures themselves, we shall require a lemma.

Lemma 6.1. *Let t be an integer with $t \geq 3$. Suppose that $\phi_1 \geq \frac{1}{15}$, $\phi_2 \leq 5\phi_1 - \frac{1}{3}$,*

$$U \leq \min\{M_2, PH_1H_2^{-3}, Q_2^{1/4}, Q_2^{5/6}M_2^{-19/6}\}, \tag{6.1}$$

and

$$Z = PU^{1-1/t}\left(P^{1/3}M_1^{2t-10-\mu_t}\right)^{1/t}. \tag{6.2}$$

Then

$$\int_0^1 |F_2(\alpha)f_2(\alpha)^{12}|d\alpha \ll P^{1+\varepsilon}\tilde{M}_2\tilde{H}_2\left(Z^{-1/4}Q_2^{\lambda_6^*} + Q_2^{\frac{3}{4}\lambda_8 - \frac{5}{4}}\right).$$

Proof. On using standard Weyl differencing, we have

$$|F_2(\alpha)|^2 \ll P(\tilde{M}_2\tilde{H}_2)^2 + \tilde{M}_2\tilde{H}_2|G(\alpha)|, \tag{6.3}$$

where

$$G(\alpha) = \sum_{\mathbf{h}}\sum_{h \leq P_2} J(\alpha),$$

and

$$J(\alpha) = \sum_{\mathbf{m}}\sum_{0<z\leq P_2-h} e\left(\frac{\alpha}{32}\Psi_3(2z+h; 2\mathbf{h}, h; \mathbf{m}, 1)\right).$$

Here the summations are over $\mathbf{m}$ and $\mathbf{h}$ satisfying (2.1). But by (3.1),

$$|J(\alpha)| = K(\alpha; \mathbf{h}, h)L_1(\alpha; \mathbf{h}, h)L_2(\alpha; \mathbf{h}, h),$$

where

$$K(\alpha; \mathbf{h}, h) = \left|\sum_{0<z\leq P_2-h} e\left(60\alpha hh_1h_2(2z+h)^2\right)\right|,$$

and for $i = 1, 2$,

$$L_i(\alpha; \mathbf{h}, h) = \left|\sum_{m_i} e\left(80\alpha hh_1h_2h_i^2m_i^{10}\right)\right|.$$

Write $\mathcal{C}(M)$ for $\mathcal{A}(MR, R) \cap (M, MR]$. Recalling (4.1) and (4.2), we find that by Hölder's inequality, we have

$$G(\alpha) \ll D(\alpha)^{1/2}E_1(\alpha)^{\frac{1}{2t}}E_2(\alpha)^{\frac{1}{2}-\frac{1}{2t}}, \tag{6.4}$$

where

$$D(\alpha) = \sum_{\mathbf{h},h} K(\alpha; \mathbf{h}, h)^2 \ll D_2(\alpha; P, \boldsymbol{\phi}),$$

$$E_1(\alpha) = \sum_{\mathbf{h},h} L_1(\alpha; \mathbf{h}, h)^{2t} \ll P^{\varepsilon}E_t(\alpha; 80H_2P_2, 2H_1, M_1; \mathcal{C}(M_1)),$$

$$E_2(\alpha) = \sum_{\mathbf{h},h} L_2(\alpha; \mathbf{h}, h)^{2+\frac{2}{t-1}} \ll P^{\varepsilon}(M_2R)^{\frac{2}{t-1}}E_1(\alpha; 160H_1P_2, H_2, M_2; \mathcal{C}(M_2)).$$

We now recall Definition 4.9. Suppose that $\alpha \in \mathfrak{m}_2$. By Dirichlet's theorem there exist $b \in \mathbb{Z}$ and $r \in \mathbb{N}$ with

$$(b,r)=1, \quad r \le P^{-1}Q_2^5 \quad \text{and} \quad |\alpha r - b| \le PQ_2^{-5}. \tag{6.5}$$

On noting that our assumptions on ϕ imply that $P \le P^{-1}Q_2^5$, we deduce from Lemma 4.1 that

$$D(\alpha) \ll P^{\varepsilon}\left(\frac{Q_2^5}{r+Q_2^5|\alpha r-b|}+P^{-1}Q_2^5\right).$$

But $\alpha \in \mathfrak{m}_2$, so either $r > P$ or $Q_2^5|\alpha r - b| \gg PR^{-15}$, and hence

$$D(\alpha) \ll P^{\varepsilon-1}Q_2^5 \ll P^{2+\varepsilon}\tilde{H}_2. \tag{6.6}$$

Next we observe that our hypotheses on ϕ imply that

$$P_2H_2M_1^{10}H_1^{-3} \ge P^{-1}(M_1^5M_2^{-1})^5 \ge P^{2/3}, \quad \text{and} \quad M_1^{10} \ge P^{1/3}.$$

Then we may apply Lemma 4.6, with $Y = P^{1/3}$ and $X = P^{-1}Q_2^5$, to deduce that

$$\begin{aligned} E_1(\alpha) &\ll P^{1+\varepsilon}\tilde{H}_2M_1^{\mu_t+10}\left((r+Q_2^5|\alpha r-b|)^{-1/3}+P^{-1/3}\right)\\ &\ll P^{2/3+\varepsilon}\tilde{H}_2M_1^{\mu_t+10}. \end{aligned} \tag{6.7}$$

Finally, since $U \le M_2$, we have $U^5 \le P$, and hence

$$U^5 \le P^{-1}Q_2^5 \le U^{-5}Q_2^5.$$

Then by Lemma 4.3, we have

$$\begin{aligned} E_2(\alpha) \ll& P^{1+\varepsilon}\tilde{H}_2M_2^{2+\frac{2}{t-1}}\left((r+Q_2^5|\alpha r-b|)^{-1/5}+U^{-1}\right)\\ \ll& P^{1+\varepsilon}\tilde{H}_2M_2^{2+\frac{2}{t-1}}U^{-1}. \end{aligned} \tag{6.8}$$

Thus, by (6.3), (6.4), (6.6), (6.7) and (6.8), we have

$$\sup_{\alpha\in\mathfrak{m}_2}|F_2(\alpha)| \ll P^{1+\varepsilon}\tilde{H}_2\tilde{M}_2Z^{-1/4}. \tag{6.9}$$

Now suppose that $\alpha \in \mathfrak{M}_2$. By Dirichlet's theorem there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a,q)=1$ and satisfying (4.24). Then since $\alpha \notin \mathfrak{m}_2$, such a and q exist with $0 \le a \le q \le P$. Thus, by Lemma 4.7 we have

$$F_2(\alpha) \ll F_2^*(\alpha) + P^{2/3+\varepsilon}\tilde{H}_2\tilde{M}_2, \tag{6.10}$$

where $F_2^*(\alpha)$ is defined as in Definition 4.9(iii). Our hypotheses on t and ϕ imply that

$$Z^{1/4} \le \left(P^{1+\frac{1}{3t}}M_2\right)^{1/4} \le P^{1/3},$$

and so by (6.9) and (6.10) we deduce that

$$\int_0^1 |F_2(\alpha)f_2(\alpha)^{12}|d\alpha \ll P^{1+\varepsilon}\tilde{H}_2\tilde{M}_2 Z^{-1/4}Q_2^{\lambda_6^*} + I, \tag{6.11}$$

where

$$I = \int_{\mathfrak{M}_2} |F_2^*(\alpha)f_2(\alpha)^{12}|d\alpha.$$

But by Hölder's inequality,

$$I \ll J_1^{3/4}J_2^{1/4}, \tag{6.12}$$

where

$$J_1 = \int_0^1 \left|f_2(\alpha)^{16}\right| d\alpha, \quad \text{and} \quad J_2 = \int_{\mathfrak{M}_2} |F_2^*(\alpha)|^4 d\alpha.$$

We have $J_1 \ll Q_2^{\lambda_8+\varepsilon}$, and by Lemma 4.10 we have $J_2 \ll P^\varepsilon(P\tilde{H}_2\tilde{M}_2)^4 Q_2^{-5}$. The lemma now follows by (6.11) and (6.12).

We are now in a position to describe the iterative processes when $s = 7$ and 8. As in §5, we let (λ_s) be an iterate of the sequence converging to (λ_s^*), and we write θ for ϕ_1 and ϕ for ϕ_2.

(i) $s = 7$.

By Lemma 6.1 we have

$$\int_0^1 |F_2(\alpha)f_2(\alpha)^{12}|d\alpha \ll P^\varepsilon(U_1 + U_2), \tag{6.13}$$

where

$$U_1 = P\tilde{M}_2\tilde{H}_2 Z^{-1/4}Q_2^{\lambda_6^*}, \tag{6.14}$$

$$U_2 = P\tilde{M}_2\tilde{H}_2 Q_2^{\frac{3}{4}\lambda_8 - \frac{5}{4}}, \tag{6.15}$$

and we must take $t \geq 3$, $\phi_1 \geq \frac{1}{15}$, and $\phi_2 \leq 5\phi_1 - \frac{1}{3}$. Here we take Z to be as large as is consistent with the conditions of Lemma 6.1. Suitable values of μ_t may be obtained by means of Lemma 3.2 of Wooley [**13**]. Using these values, it transpires that a good choice for t is 22, and we may take $\mu_{22} = 34.228489$.

For the moment, suppose that our ultimate choices for θ and ϕ imply that U_1 is the dominating contribution. Then proceeding as described in §2, using the iterative sequence for $s = 7$ given above, the equations for λ_7, θ, and ϕ are determined by

$$PH_1M_1M_2Q_2^{\lambda_6^*} \approx PM_1M_2H_1H_2Z^{-1/4}Q_2^{\lambda_6^*}, \tag{6.16}$$

$$PM_1Q_1^{\lambda_6^*} \approx \left(P(M_1H_1)^2M_2^{12}Q_2^{\lambda_6^*}Q_1^{\lambda_6^*}\right)^{1/2}, \tag{6.17}$$

$$P^{\lambda_7} \approx PM_1^{12}Q_1^{\lambda_6^*}. \tag{6.18}$$

Write $\Delta = \lambda_6^* - 7$, and $\delta = \mu_{22} - 34$. Suppose now that our ultimate choices for θ and ϕ imply that (6.1) holds when

$$U = Q_2^{5/6} M_2^{-19/6}. \tag{6.19}$$

The equations (6.2), (6.16) and (6.17) then yield

$$\begin{aligned} 4 - 20\phi =& 1 + \frac{21(5 - 24\phi - 5\theta)}{132} + \frac{(1 - 3\delta\theta)}{66}, \\ 10\theta =& 1 + (5 - \Delta)\phi. \end{aligned}$$

Therefore

$$\phi = \frac{289 + 105\theta + 6\delta\theta}{2136},$$

and hence

$$\theta = \frac{3581 - 289\Delta}{20835 + 105\Delta - 6\delta(5 - \Delta)}. \tag{6.20}$$

Calculating θ and ϕ, we find that $\theta \le 0.163961$ and $\phi \le 0.143465$. A simple calculation now shows that our choices for U and Z were indeed justified.

We must now check that U_1 is indeed the dominating contribution. This will follow from (6.14) and (6.15) provided that

$$Z^{-1/4} Q_2^{\lambda_6^*} \gg Q_2^{\frac{1}{4}(3\lambda_8 - 5)}.$$

This inequality holds provided that

$$(4\lambda_6^* + 5 - 3\lambda_8)(1 - \theta - \phi) > 1 + \frac{21(5 - 24\phi - 5\theta)}{132} + \frac{1 - 3\delta\theta}{66}. \tag{6.21}$$

In order to check that the condition (6.21) is satisfied, we shall plainly require a suitable estimate for λ_8. We can, however, make do with a relatively poor estimate, and to this end we will make use of inequality $(k - 2)$ of §4 of Vaughan [**8**]. Thus it suffices to use the iterates

$$\lambda_7 = \tfrac{34}{41}\lambda_6^* + \tfrac{125}{41}, \quad \lambda_8 = \tfrac{34}{41}\lambda_7 + \tfrac{139}{41},$$

whence we deduce that $\lambda_8 \le 11.10486$. In view of our choices for θ and ϕ, this is enough to show that U_1 is indeed the dominating contribution.

Since the value of θ given by (6.20) is independent of λ_s with $s > 6$, we deduce from (6.18) that

$$\lambda_7^* = \lambda_6^*(1 - \theta) + 1 + 12\theta.$$

Thus we obtain $\lambda_7^* \le 9.272729$.

(ii) $s = 8$.

Initially, we may proceed precisely as in case (i), using the estimate (6.13). For the moment, suppose that our ultimate choices for θ and ϕ imply that U_1 is the dominating contribution. Then proceeding as described in §2, using the iterative sequence for $s = 8$ given above, the equations for λ_8, θ and ϕ are determined by

$$PH_1M_1M_2Q_2^{\lambda_6^*} \approx PM_1M_2H_1H_2Z^{-1/4}Q_2^{\lambda_6^*}, \tag{6.22}$$
$$PM_1Q_1^{\lambda_7^*} \approx \Big(P(M_1H_1)^2M_2^{12}Q_2^{\lambda_6^*}Q_1^{\lambda_8}\Big)^{1/2}, \tag{6.23}$$
$$P^{\lambda_8} \approx PM_1^{14}Q_1^{\lambda_7^*}. \tag{6.24}$$

Write $\mathcal{E} = \lambda_8 - 2\lambda_7^* + \lambda_6^*$. Also, as in case (i), write $\Delta = \lambda_6^* - 7$ and $\delta = \mu_{22} - 34$. Suppose that our ultimate choices for θ and ϕ imply that (6.1) holds when U satisfies (6.19). The equations (6.2), (6.22) and (6.23) then yield

$$4 - 20\phi = 1 + \frac{21(5 - 24\phi - 5\theta)}{132} + \frac{1 - 3\delta\theta}{66},$$
$$10\theta = 1 + \mathcal{E}(1 - \theta) + (5 - \Delta)\phi.$$

Therefore

$$\phi = \frac{289 + 105\theta + 6\delta\theta}{2136}, \tag{6.25}$$

and hence

$$\theta = \frac{3581 + 2136\mathcal{E} - 289\Delta}{20835 + 2136\mathcal{E} + 105\Delta - 6\delta(5 - \Delta)}. \tag{6.26}$$

Given an iterate for λ_8, we therefore obtain the next iterate as follows. We compute θ and ϕ from (6.25) and (6.26). We then check that the choice of U given by (6.19) is indeed permissible, and check that U_1 is the dominating contribution. The latter follows provided that (6.21) holds. The next iterate for λ_8 is then given by (6.18), that is, by

$$\lambda_8' = \lambda_7^*(1 - \theta) + 1 + 14\theta. \tag{6.27}$$

To succeed with this iteration process, we need to start with an initial iterate for λ_8 reasonably close to λ_8^*. For this purpose we can use inequality $(k-2)$ of §4 of Vaughan [**8**] once again. We therefore take

$$\lambda_8 = \tfrac{34}{41}\lambda_7^* + \tfrac{139}{41}.$$

A computation now shows that $\lambda_8^* \le 11.077363$. We note that λ_8^* can be calculated directly as the larger root of the quadratic equation obtained by eliminating θ between (6.26) and (6.27), equating λ_8' and λ_8 and recalling that λ_8 occurs linearly in $\mathcal{E}$.

We summarise in the Appendix the converged values of $\boldsymbol{\lambda}^*$ as computed to 15 significant figures and rounded up in the last figure displayed.

7. The proof of Theorem 1.1 for fifth powers

We shall prove Theorem 1.1 for fifth powers by using a variant of the Hardy-Littlewood method. In this section our notational demands are somewhat different. We suppose that ε, η and τ are sufficiently small positive numbers, with η and τ depending at most on ε, and ε and η, respectively. In addition, we suppose that n is sufficiently large in terms of ε, η, and τ. We adopt the convention that whenever δ appears in a statement, then the statement holds for some positive number δ independent of n. Write

$$P = n^{1/5}, \quad R = P^{\eta}, \quad \sigma = \tfrac{1}{16}, \quad \text{and} \quad \theta = \tfrac{7}{41}.$$

We let $M_1, \dots, M_8$ be real numbers satisfying

$$P^{\theta} \le M_s \le P^{\theta+\tau}, \tag{7.1}$$

and for convenience write

$$Q_s = PM_s^{-1} \text{ and } H_s = PM_s^{-5}.$$

Consider the number $r(n;\mathbf{M}) = r(n; M_1, \dots, M_8)$ of solutions of the equation

$$x^5 + y^5 + x_1^5 + \dots + x_7^5 + p_1^5 y_1^5 + \dots + p_8^5 y_8^5 = n, \tag{7.2}$$

with the p_s primes satisfying

$$p_s \equiv -1 \pmod 5, \quad M_s < p_s \le 2M_s, \tag{7.3}$$

and with

$$1 \le x, y \le P, \quad x_j \in \mathcal{A}(P,R) \quad (1 \le j \le 7),$$
$$y_s \in \mathcal{A}(Q_s, R) \quad (1 \le s \le 8).$$

We shall show that

$$\sum_{M_1} \cdots \sum_{M_8} r(n;\mathbf{M}) \gg n^{12/5}, \tag{7.4}$$

where the multiple sum is over all choices of M_s of the form

$$M_s = 2^u P^{\theta}, \tag{7.5}$$

and satisfying (7.1). Since $p_s > R$, each solution of (7.2) gives rise to a unique representation of n as the sum of 17 fifth powers of positive integers in the sense that the ordered 17-tuple $x, y, x_1, \dots, x_7, p_1 y_1, \dots, p_8 y_8$ is unique. Hence the verification of (7.4) is sufficient to establish Theorem 1.1 when $k = 5$.

We henceforth assume that the M_s are of the form (7.5). Let

$$F(\alpha) = \sum_{1 \le x \le P} e(\alpha x^5), \quad g_s(\alpha) = \sum_{x \in \mathcal{A}(Q_s, R)} e(\alpha x^5),$$

$$f(\alpha) = \sum_{x \in \mathcal{A}(P,R)} e(\alpha x^5), \quad h_s(\alpha) = \sum_{p_s} g_s(\alpha p_s^5),$$

where the p_s satisfy (7.3). Then

$$r(n; \mathbf{M}) = \int_0^1 \mathcal{F}_1(\alpha)\mathcal{F}_2(\alpha)e(-\alpha n)d\alpha, \tag{7.6}$$

where

$$\mathcal{F}_1(\alpha) = F(\alpha)f(\alpha)^7, \tag{7.7}$$

$$\mathcal{F}_2(\alpha) = F(\alpha)\prod_{s=1}^{8} h_s(\alpha). \tag{7.8}$$

Let $C = 25 \cdot 3^{25}$, $M = P^{\theta+\tau}$, and $Q = PM^{-1}$. Write

$$\mathcal{I} = (C^{-1}P^{1-\sigma}Q^{-5}, 1 + C^{-1}P^{1-\sigma}Q^{-5}].$$

Let $\mathfrak{m}$ denote the set of real numbers α in $\mathcal{I}$ with the property that whenever

$$a \in \mathbb{Z}, \quad q \in \mathbb{N}, \quad (a,q) = 1 \quad \text{and} \quad |\alpha q - a| \le C^{-1}P^{1-\sigma}Q^{-5},$$

then one has $q > P^{1-\sigma}M^5$. Let $\mathfrak{M}$ denote the major arcs $\mathcal{I} \setminus \mathfrak{m}$; that is, the union of the intervals

$$\mathfrak{M}(q,a) = \{\alpha : |\alpha q - a| \le C^{-1}P^{1-\sigma}Q^{-5}\}$$

with $1 \le a \le q \le P^{1-\sigma}M^5$ and $(a,q) = 1$.

We first consider the minor arcs $\mathfrak{m}$.

Lemma 7.1. *We have*

$$\int_{\mathfrak{m}} \mathcal{F}_1(\alpha)\mathcal{F}_2(\alpha)e(-\alpha n)d\alpha \ll P^{12-\delta}.$$

Proof. By Schwarz's inequality we have

$$\left|\int_{\mathfrak{m}} \mathcal{F}_1(\alpha)\mathcal{F}_2(\alpha)e(-\alpha n)d\alpha\right| \le \left(\int_0^1 |\mathcal{F}_1(\alpha)|^2 d\alpha\right)^{1/2} \left(\int_{\mathfrak{m}} |\mathcal{F}_2(\alpha)|^2 d\alpha\right)^{1/2}. \tag{7.9}$$

The first integral on the right-hand side is

$$\int_0^1 |F(\alpha)^2 f(\alpha)^{14}| d\alpha \ll P^{\lambda_8+\varepsilon}, \tag{7.10}$$

by using the conclusions of §6 (see, for example, the note at the end of §3 of Wooley [**13**]). Also, by the argument of Lemma 3.2 of Vaughan and Wooley [**10**], we have

$$\int_0^1 \left(\sum_{p_s} |g(\alpha p_s^5)|\right)^{16} d\alpha \ll M_s P^{\lambda_8^+ + 5\tau + \varepsilon},$$

where

$$\lambda_8^+ = \lambda_7(1-\theta) + 1 + 14\theta. \tag{7.11}$$

Note in particular that $\lambda_8^+ \le 11.079825$. Using this estimate, we may follow through the argument of §4 of Vaughan and Wooley [**10**] to obtain

$$\int_{\mathfrak{m}} |\mathcal{F}_2(\alpha)|^2 d\alpha \ll \prod_{s=1}^{8} \left(M_s^{15} I_s + J_s\right)^{1/8},$$

where

$$J_s \ll Q_s^{2-2\sigma+2\varepsilon} M_s P^{\lambda_8^+ + 5\tau+\varepsilon},$$

and

$$I_s \ll \left(PM_s + (PM_s)^{1-2\sigma+\varepsilon} H_s\right) Q_s^{\lambda_8+\varepsilon}.$$

A little computation reveals that

$$\int_{\mathfrak{m}} |\mathcal{F}_2(\alpha)|^2 d\alpha \ll P^{13-\delta_1}, \tag{7.12}$$

with $\delta_1 > 0.082$. The lemma now follows on combining (7.7)-(7.10) and (7.12).

We now consider the major arcs $\mathfrak{M}$. Let

$$v(\beta) = \sum_{1\le x\le n} \tfrac{1}{5} x^{-4/5} e(\beta x),$$

and

$$S(q,a) = \sum_{r=1}^{q} e(ar^5/q).$$

Define $V(\alpha)$ on $\mathfrak{M}$ by taking

$$V(\alpha) = q^{-1} S(q,a) v(\alpha - a/q)$$

whenever $\alpha \in \mathfrak{M}(q,a)$. Since the $\mathfrak{M}(q,a)$ with $1 \le a \le q \le P^{1-\sigma}M^5$ are disjoint, it follows that $V(\alpha)$ is well-defined.

Lemma 7.2. *We have*

$$\sum_{\mathbf{M}} r(n;\mathbf{M}) = \int_{\mathfrak{M}} V(\alpha)^2 f(\alpha)^7 \left(\prod_{s=1}^{8} \sum_{M_s} h_s(\alpha)\right) e(-\alpha n) d\alpha + O(P^{12-\delta}).$$

Proof. Write $\Delta(\alpha) = F(\alpha) - V(\alpha)$. Then by Theorem 2 of Vaughan [**4**], we have

$$\Delta(\alpha) \ll q^{\varepsilon} \left(q + P^5|\alpha q - a|\right)^{1/2} \quad (\alpha \in \mathfrak{M}(q,a)). \tag{7.13}$$

Hence, for $\alpha \in \mathfrak{M}$ we have $\Delta(\alpha) \ll P^{2\varepsilon}(P^{1-\sigma}M^5)^{1/2}$. Then by Schwarz's inequality,

$$\int_{\mathfrak{M}} \left|\Delta(\alpha)^2 f(\alpha)^7 h_s(\alpha)^8\right| d\alpha \ll P^{1-\sigma+4\varepsilon} M^5 J_1^{1/2} J_2^{1/2}, \tag{7.14}$$

where

$$J_1 = \int_0^1 |f(\alpha)|^{14} d\alpha, \quad \text{and} \quad J_2 = \int_0^1 |h_s(\alpha)|^{16}\, d\alpha.$$

By the conclusions of the previous section,

$$J_1 \ll P^{\lambda_7+\varepsilon}. \tag{7.15}$$

Also, it follows by the argument of Lemma 3.1 of Vaughan and Wooley [**10**] that

$$J_2 \ll P^{\lambda_8^+ + 5\tau+\varepsilon}, \tag{7.16}$$

where λ_8^+ is given by (7.11). Then the right hand side of (7.14) is

$$\ll P^{1-\sigma+4\varepsilon} M^5 \left(P^{\lambda_7+\varepsilon} P^{\lambda_8^+ + 5\tau+\varepsilon}\right)^{1/2} \ll P^{12-\delta}.$$

Next, by appealing to Lemma 4.6 of Vaughan [**5**], we obtain

$$V(\alpha) \ll P(q + P^5|\alpha q - a|)^{-1/5} \quad (\alpha \in \mathfrak{M}(q,a)),$$

and hence, by (7.13),

$$V(\alpha)\Delta(\alpha) \ll P^{1+2\varepsilon}(P^{1-\sigma}M^5)^{\frac{3}{10}} \quad (\alpha \in \mathfrak{M}(q,a)).$$

Therefore, as above, we obtain

$$\int_{\mathfrak{M}} \left|V(\alpha)\Delta(\alpha) f(\alpha)^7 h_s(\alpha)^8\right| d\alpha \ll P^{1+2\varepsilon}(P^{1-\sigma}M^5)^{\frac{3}{10}} J_1^{1/2} J_2^{1/2} \ll P^{12-\delta}, \tag{7.17}$$

by (7.15) and (7.16). Collecting together (7.6)-(7.8), (7.14), (7.17), and Lemma 7.1, the proof of the lemma is completed.

Before proceeding to estimate the contribution of the major arcs, we establish an auxiliary lemma. Let

$$K_1 = \int_0^1 |f(\alpha)|^{18} d\alpha \quad \text{and} \quad K_2^{(s)} = \int_0^1 \left|\sum_{M_s} h_s(\alpha)\right|^{18} d\alpha. \tag{7.18}$$

Lemma 7.3. *We have*

$$K_1 \ll P^{13} \quad \textit{and} \quad K_2^{(s)} \ll P^{13} \quad (1 \le s \le 8).$$

Proof. Write $f_1(\alpha)$ for $f(\alpha)$, and $f_2(\alpha)$ for $\sum_{M_s} h_s(\alpha)$. Let

$$F_i(\alpha) = \sum_{1 \le x \le P_i} e(\alpha x^5) \quad (i = 1, 2),$$

with $P_i = 2^{i-1}P$. Also, for the sake of convenience, write K_2 for $K_2^{(s)}$. Then by considering the underlying diophantine equations, we have for $i = 1, 2$,

$$K_i \le \int_0^1 |F_i(\alpha)^2 f_i(\alpha)^{16}| d\alpha.$$

We apply the Hardy-Littlewood method. Define

$$\mathfrak{W}(q, a) = \{\alpha : |q\alpha - a| \le \tfrac{1}{10} P_i^{-4}\}$$

for $1 \le a \le q \le P$ and $(a, q) = 1$, and define $\mathfrak{W}$ to be the union of these arcs, and $\mathfrak{w} = (\frac{1}{10}P_i^{-4}, 1 + \frac{1}{10}P_i^{-4}] \setminus \mathfrak{W}$. Then by Weyl's inequality, we have $\sup_{\alpha \in \mathfrak{w}} |F_i(\alpha)| \ll P^{1-\sigma+\varepsilon}$. Hence

$$K_i \ll P^{2-2\sigma+2\varepsilon} \cdot P^{\lambda_8^+ + \varepsilon} + \int_{\mathfrak{W}} |F_i(\alpha)^2 f_i(\alpha)^{16}| d\alpha, \tag{7.19}$$

where λ_8^+ (which satisfies $\lambda_8^+ > \lambda_8$) is given by (7.11). By using Lemma 5.1 of Vaughan [**8**] combined with Hölder's inequality, we deduce that

$$\begin{aligned}\int_{\mathfrak{W}} |F_i(\alpha)^2 f_i(\alpha)^{16}| d\alpha \ll & \left(\int_{\mathfrak{W}} |F_i(\alpha)|^{18} d\alpha \right)^{1/9} \left(\int_0^1 |f_i(\alpha)|^{18} d\alpha \right)^{8/9} \\ \ll & P^{13/9} K_i^{8/9}.\end{aligned}$$

Then by (7.19),

$$K_i \ll P^{13-\delta} + P^{13/9} K_i^{8/9},$$

and hence $K_i \ll P^{13}$, which completes the proof of the lemma.

We now attend to the matter of pruning the major arcs. Let W denote a parameter to be chosen later, and let $\mathfrak{N}$ denote the union of the intervals

$$\mathfrak{N}(q, a) = \{\alpha : |\alpha q - a| \le W P^{-5}\},$$

with $(a, q) = 1$ and $1 \le a \le q \le W$. We assume that $1 \le W \le P^{1/2}$, so that $\mathfrak{N} \subset \mathfrak{M}$. Let $\mathfrak{P} = \mathfrak{M} \setminus \mathfrak{N}$.

By Hölder's inequality combined with the methods of §4.4 of Vaughan [**5**] (cf. Lemma 5.1 of Vaughan [**8**]), we obtain, on recalling (7.18),

$$\begin{aligned}\int_{\mathfrak{P}} \left| V(\alpha)^2 f(\alpha)^7 \prod_{s=1}^{8} \left(\sum_{M_s} h_s(\alpha p_s^5) \right) \right| d\alpha &\ll K_1^{\frac{7}{18}} \left(\prod_{s=1}^{8} K_2^{(s)} \right)^{\frac{1}{18}} \left(\int_{\mathfrak{P}} |V(\alpha)|^{12} d\alpha \right)^{\frac{1}{6}} \\ &\ll (P^{13})^{5/6} (P^7 W^{-6\delta})^{1/6} \\ &\ll P^{12} W^{-\delta}.\end{aligned}$$

By the methods of §5 of Vaughan [**8**], when $W \le \log P$, $q \le \log P$, and $(a,q) = 1$, we have

$$\sum_{p_s} g_s(\alpha p_s^5) = q^{-1} S(q,a) u_s(\alpha - a/q) + O\left(\frac{P}{\log P} (q + P^5 |\alpha q - a|) \right)$$

where

$$u_s(\beta) = \sum_{x \le (2P)^5} \frac{\min\{\log(2Px^{-1/5}), \log 2\}}{4 \log M_s} \tfrac{1}{5} x^{-4/5} \rho \left(\frac{\log(x^{1/5}/M_s)}{\log R} \right) e(\beta x),$$

and $\rho(x)$ is Dickman's function, defined for real x by

$$\rho(x) = 0 \text{ when } x \le 0,$$

$$\rho(x) = 1 \text{ when } 0 < x \le 1,$$

$$\rho \text{ is continuous for } x > 0,$$

$$\rho \text{ is differentiable for } x > 1,$$

$$x\rho'(x) = -\rho(x-1) \text{ for } x > 1.$$

Also, by Lemma 5.4 of Vaughan [**8**], we have

$$f(\alpha) = q^{-1} S(q,a) w(\alpha - a/q) + O\left(\frac{P}{\log P} (q + P^5 |\alpha q - a|) \right)$$

and

$$w(\beta) \ll P(1 + P^5 \|\beta\|)^{-1/5},$$

where

$$w(\beta) = \sum_{R^5 < m \le n} \tfrac{1}{5} m^{-4/5} \rho \left(\frac{\log m}{5 \log R} \right) e(\beta m).$$

Then as in §5 of Vaughan and Wooley [**10**], we deduce that when ϕ is sufficiently small, and $W = (\log P)^{\phi}$, then we have

$$\int_{\mathfrak{N}} V(\alpha)^2 f(\alpha)^7 \left(\prod_{s=1}^{8} h_s(\alpha) \right) e(-\alpha n) d\alpha = \mathfrak{S}(n) J(n) + O\left(P^{12} (\log P)^{-8-\delta} \right)$$

where $\mathfrak{S}(n)$ is the usual singular series in Waring's problem,

$$\mathfrak{S}(n)=\sum_{q=1}^{\infty}\sum_{\substack{a=1\\(a,q)=1}}^{q}\left(q^{-1}S(q,a)\right)^{17}e(-an/q),$$

and

$$J(n)=\int_0^1 v(\beta)^2w(\beta)^7\left(\prod_{s=1}^{8}u_s(\beta)\right)e(-\beta n)d\beta.$$

Now by Theorem 4.6 of Vaughan [**5**], we have $1\ll\mathfrak{S}\ll 1$, and a simple counting argument shows that $J(n)\gg n^{12/5}(\log n)^{-8}$. Thus

$$\begin{aligned}\sum_{\mathbf{M}}r(n;\mathbf{M})&=\sum_{\mathbf{M}}\int_{\mathfrak{N}}V(\alpha)^2f(\alpha)^7\left(\prod_{s=1}^{8}h_s(\alpha)\right)e(-\alpha n)d\alpha+O\left(P^{12}(\log P)^{-\delta}\right)\\&\gg n^{12/5},\end{aligned}$$

and this completes the proof of Theorem 1.1 for fifth powers.

8. The iterative schemes for $k\geq 6$: second differences

In the remainder of this paper we shall restrict attention to those k with $6\leq k\leq 9$. As usual, for $s=1$ and 2 we have the classical bounds

$$S_s(P,R)\ll P^{s+\varepsilon},$$

and for $s=3$ and 4 we use the results of Theorem 1.4 of Vaughan [**9**]. These give

$$\lambda_3^*=3+2\theta\quad\text{and}\quad\lambda_4^*=\frac{4+(k-3)\theta}{1-\theta},$$

where θ is the smallest non-negative root of the polynomial

$$3-(3k^2-(e+11)k+e+22)\theta-(k(e+15)-3e-48)\theta^2-(2k+2e+32)\theta^3$$

and

$$\begin{cases}e=0, & \text{when } k=6,7,9;\\ e=1, & \text{when } k=8.\end{cases}$$

The values of λ_3^* and λ_4^* obtained in this way are listed in the Appendix.

In what follows, we let (λ_s) be an iterate of the sequence converging to (λ_s^*), and we write $\theta=\phi_1$ and $\phi=\phi_2$. Note that to obtain a reasonable initial iterate (λ_s), we may use the values given by Lemma 3.2 of Wooley [**13**]. Our argument divides into cases according to the values of s and k.

(i) $s=5$ and $k=6,7,8$.

In these cases we adopt the iterative procedure displayed below.

$$\begin{array}{ccccccc} F_0^2 f_0^8 & \longmapsto F_1 f_1^8 & \longrightarrow & F_2 f_2^6 & \Longrightarrow & (F_2^2)^{\frac{3}{10}}(F_2^4)^{\frac{1}{10}}(f_2^{10})^{\frac{3}{5}} \\ & & & \downarrow & & \\ & & & f_1^{10} & & \end{array}$$

Let

$$\begin{cases} e = 0, & \text{when } k = 8; \\ e = 1, & \text{when } k = 6, 7. \end{cases}$$

Then by Theorem 3.10(Ib) case (i) when $k = 6, 7$, and Theorem 3.10(II) when $k = 8$, we have

$$\int_0^1 |F_2(\alpha)|^2 d\alpha \ll P^{1+\varepsilon} \tilde{H}_2 \tilde{M}_2^{2-e}. \tag{8.1}$$

Also, by Theorem 3.11(I) case (i) when $k = 6, 7$, and Theorem 3.11(II) when $k = 8$, we have

$$\int_0^1 |F_2(\alpha)|^4 d\alpha \ll P^{2+\varepsilon} \tilde{H}_2^3 \tilde{M}_2^{4-e}. \tag{8.2}$$

Then proceeding as described in §2, using the iterative sequence for $s = 5$ given above, the equations for λ_5, θ and ϕ are determined by

$$PH_1M_1M_2Q_2^{\lambda_3^*} \approx P^{1/2}(H_1H_2)^{3/5}(M_1M_2)^{1-\frac{2}{5}e}\left(Q_2^{\lambda_5}\right)^{3/5}, \tag{8.3}$$

$$PM_1Q_1^{\lambda_4^*} \approx \left(P(M_1H_1)^2 M_2^6 Q_2^{\lambda_3^*} Q_1^{\lambda_5}\right)^{1/2}, \tag{8.4}$$

$$P^{\lambda_5} \approx PM_1^8 Q_1^{\lambda_4^*}. \tag{8.5}$$

On writing $\delta = 6\lambda_5 - 10\lambda_3^*$, equation (8.3) leads to the equation

$$\delta(1-\theta-\phi) + 4(k-e)\theta - 3 - (6k+4e)\phi = 0,$$

and hence

$$\phi = \frac{4(k-e)\theta + \delta(1-\theta) - 3}{6k+4e+\delta}. \tag{8.6}$$

Meanwhile, equation (8.4) leads to the equation

$$2(1+\theta+\lambda_4^*(1-\theta)) = \lambda_5(1-\theta) + \lambda_3^*(1-\theta-\phi) + 3 - (2k-2)\theta + 6\phi.$$

On writing $\mathcal{E} = \lambda_5 - 2\lambda_4^* + \lambda_3^*$, we obtain

$$\mathcal{E}(1-\theta) + 1 - 2k\theta = (\lambda_3^* - 6)\phi. \tag{8.7}$$

Write

$$\alpha = \frac{\lambda_3^* - 6}{6k+4e+\delta}.$$

Then (8.6) and (8.7) yield

$$\theta = \frac{1 + \mathcal{E} + \alpha(3-\delta)}{2k + \mathcal{E} + \alpha(4(k-e)-\delta)}.$$

From (8.5), the next iterate for λ_5 is therefore given by

$$\lambda_5' = \lambda_4^*(1-\theta) + 1 + 8\theta.$$

The above iteration process converges to λ_5^*, with

$$\lambda_5^* = \lambda_4^*(1-\theta_5) + 1 + 8\theta_5, \tag{8.8}$$

where θ_5 is a root of the cubic equation obtained by substituting the expression (8.8) into

$$(2k + \mathcal{E}^* + \alpha^*(4(k-e)-\delta^*))\theta_5 = 1 + \mathcal{E}^* + \alpha^*(3-\delta^*),$$

with

$$\delta^* = 6\lambda_5^* - 10\lambda_3^*, \quad \mathcal{E}^* = \lambda_5^* - 2\lambda_4^* + \lambda_3^*, \quad \alpha^* = \frac{\lambda_3^* - 6}{6k + 4e + \delta^*}.$$

The values of λ_5^* obtained in this way are listed in the Appendix.

(ii) $s = 6$ and $k = 6, 7, 8$.

In these cases we adopt the iterative procedure displayed below.

$$\begin{array}{ccccccc} F_0^2 f_0^{10} & \longmapsto & F_1 f_1^{10} & \longrightarrow & F_2 f_2^8 & \Longrightarrow & (F_2^4)^{1/4}(f_2^{10})^{1/2}(f_2^{12})^{1/4} \\ & & & & \downarrow & & \\ & & & & f_1^{12} & & \end{array}$$

We observe that the estimates (8.1) and (8.2) hold once again. Then proceeding as described in §2, using the iterative sequence for $s = 6$ given above, the equations for λ_6, θ and ϕ are determined by

$$PH_1M_1M_2Q_2^{\lambda_4^*} \approx P^{1/2}(H_1H_2)^{3/4}(M_1M_2)^{1-\frac{1}{4}e}\left(Q_2^{\lambda_5^*}\right)^{1/2}\left(Q_2^{\lambda_6}\right)^{1/4}, \tag{8.9}$$

$$PM_1Q_1^{\lambda_5^*} \approx \left(P(M_1H_1)^2M_2^8Q_2^{\lambda_4^*}Q_1^{\lambda_6}\right)^{1/2}, \tag{8.10}$$

$$P^{\lambda_6} \approx PM_1^{10}Q_1^{\lambda_5^*}. \tag{8.11}$$

On writing $\delta = 2\lambda_5^* + \lambda_6 - 4\lambda_4^*$, equation (8.9) leads to the equation

$$\delta(1-\theta-\phi) + (k-e)\theta - (3k+e)\phi = 0,$$

and hence

$$\phi = \frac{(k-e)\theta + \delta(1-\theta)}{3k+e+\delta}. \tag{8.12}$$

Meanwhile, equation (8.10) leads to the equation

$$2(1+\theta+\lambda_5^*(1-\theta)) = \lambda_6(1-\theta) + \lambda_4^*(1-\theta-\phi) + 3 - 2(k-1)\theta + 8\phi.$$

On writing $\mathcal{E} = \lambda_6 - 2\lambda_5^* + \lambda_4^*$, we obtain

$$\mathcal{E}(1-\theta) + 1 - 2k\theta = (\lambda_4^* - 8)\phi. \tag{8.13}$$

Write

$$\alpha = \frac{\lambda_4^* - 8}{3k + e + \delta}.$$

Then (8.12) and (8.13) yield

$$\theta = \frac{1 + \mathcal{E} - \alpha\delta}{2k + \mathcal{E} + \alpha(k - e - \delta)}.$$

From (8.11), the next iterate for λ_6 is therefore given by

$$\lambda_6' = \lambda_5^*(1-\theta) + 1 + 10\theta.$$

The above iteration process converges to λ_6^*, with

$$\lambda_6^* = \lambda_5^*(1-\theta_6) + 1 + 10\theta_6, \tag{8.14}$$

where θ_6 is a root of the cubic equation obtained by substituting the expression (8.14) into

$$(2k + \mathcal{E}^* + \alpha^*(k - e - \delta^*))\theta_6 = 1 + \mathcal{E}^* - \alpha^*\delta^*,$$

with

$$\delta^* = 2\lambda_5^* + \lambda_6^* - 4\lambda_4^*, \quad \mathcal{E}^* = \lambda_6^* - 2\lambda_5^* + \lambda_4^*, \quad \alpha^* = \frac{\lambda_4^* - 8}{3k + e + \delta^*}.$$

The values of λ_6^* obtained in this way are listed in the Appendix.

(iii) $s = 5, 6$ and $k = 9$.

In these cases we use the iterative procedures displayed below.

$s = 5$.

$$\begin{array}{ccccccc} F_0^2 f_0^8 & \longmapsto & F_1 f_1^8 & \longrightarrow & F_2 f_2^6 & \Longrightarrow & (F_2^2)^{1/2}(f_2^{12})^{1/2} \\ & & & & \big\downarrow & & \\ & & & & f_1^{10} & & \end{array}$$

$s = 6$.

$$\begin{array}{ccccccc} F_0^2 f_0^{10} & \longmapsto & F_1 f_1^{10} & \longrightarrow & F_2 f_2^8 & \Longrightarrow & (F_2^2)^{1/6}(F_2^4)^{1/6}(f_2^{12})^{2/3} \\ & & & & \big\downarrow & & \\ & & & & f_1^{12} & & \end{array}$$

The iterative procedure is now more complicated since both schemes depend on both λ_5 and λ_6.

By Theorem 3.10(Ib) case (i), we have estimate (8.1) with $e = 1$. Also, by Theorem 3.11(II), we have estimate (8.2) with $e = 0$. Thus we find that the initial arguments of parts (i) and (ii) of this section hold, but with (8.3) replaced by

$$PH_1M_1M_2Q_2^{\lambda_3^*} \approx P^{1/2}(H_1H_2M_1M_2)^{1/2}\left(Q_2^{\lambda_6}\right)^{1/2},$$

and (8.9) replaced by

$$PH_1M_1M_2Q_2^{\lambda_4^*} \approx P^{1/2}(H_1H_2)^{2/3}(M_1M_2)^{5/6}\left(Q_2^{\lambda_6}\right)^{2/3}.$$

Writing θ_s, ϕ_s for ϕ_1, ϕ_2 for each s, we find that the next iterates for $(\lambda_s, \theta_s, \phi_s)$ $(s = 5, 6)$ are given by

$$\lambda_5' = \lambda_4^*(1 - \theta_5) + 1 + 8\theta_5,$$

with

$$\phi_5 = \frac{(k-1)\theta_5 + \delta_5(1-\theta_5) - 1}{k + 1 + \delta_5},$$
$$\theta_5 = \frac{1 + \mathcal{E}_5 - \alpha_5(1-\delta_5)}{2k + \mathcal{E}_5 - \alpha_5(k - 1 - \delta_5)},$$
$$\mathcal{E}_5 = \lambda_5 - 2\lambda_4^* + \lambda_3^*,$$
$$\delta_5 = \lambda_6 - 2\lambda_3^*,$$
$$\alpha_5 = \frac{6 - \lambda_3^*}{k + 1 + \delta_5},$$

and

$$\lambda_6' = \lambda_5(1 - \theta_6) + 1 + 10\theta_6,$$

with

$$\phi_6 = \frac{(2k-1)\theta_6 + \delta_6(1-\theta_6) - 1}{4k + 1 + \delta_6},$$
$$\theta_6 = \frac{1 + \mathcal{E}_6 - \alpha_6(1-\delta_6)}{2k + \mathcal{E}_6 - \alpha_6(2k - 1 - \delta_6)},$$
$$\mathcal{E}_6 = \lambda_6 - 2\lambda_5 + \lambda_4^*,$$
$$\delta_6 = 4\lambda_6 - 6\lambda_4^*,$$
$$\alpha_6 = \frac{8 - \lambda_4^*}{4k + 1 + \delta_6}.$$

The converged values for λ_5^* and λ_6^* obtained in this way are listed in the Appendix.

9. The iterative process for $k \geq 6$: third differences

When $k \geq 6$ we need to make use of differences higher than the second. As usual we let (λ_s) be an iterate of the sequence converging to (λ_s^*), and to simplify formulae we write $\theta = \phi_1$, $\phi = \phi_2$, and $\psi = \phi_3$. Note also that we may use the λ_t^* already established for smaller t, and Lemma 3.2 of Wooley [**13**] to provide initial values for the λ_t under consideration.

Let

$$\begin{cases} e = 0, & \text{when } k = 9; \\ e = 1, & \text{when } k = 6, 7, 8. \end{cases}$$

By Theorem 3.10(Ib) case (i) when $k = 7, 8$, and Theorem 3.10(II) when $k = 9$, we have

$$\int_0^1 |F_3(\alpha)|^2 d\alpha \ll P^{1+\varepsilon} \tilde{H}_3 \tilde{M}_3^{2-e}. \tag{9.1}$$

Also, by Theorem 3.10(Ib) case (iii), estimate (9.1) also holds when $k = 6$ provided that

$$\theta + (k+1)(\phi + \psi) \leq 2. \tag{9.2}$$

Further, by Theorem 3.11(I) case (i) when $k = 8$, and Theorem 3.11(II) when $k = 9$, we have

$$\int_0^1 |F_3(\alpha)|^4 d\alpha \ll P^{2+\varepsilon} \tilde{H}_3^3 \tilde{M}_3^{4-e}. \tag{9.3}$$

Also, by Theorem 3.11(I) case (iii), estimate (9.3) also holds when $k = 6$ and 7 provided that inequality (9.2) is satisfied.

We divide into cases according to the values of s and k.

(i) $s = 7$ and $k = 6, 7$.

In these cases we adopt the iterative procedure displayed below.

$$\begin{array}{ccccccccc} F_0^2 f_0^{12} & \longmapsto & F_1 f_1^{12} & \longrightarrow & F_2 f_2^{10} & \longrightarrow & F_3 f_3^8 & \Longrightarrow & (F_3^2)^{\frac{1}{6}} (F_3^4)^{\frac{1}{6}} (f_3^{12})^{\frac{2}{3}} \\ & & & & \big\downarrow & & \big\downarrow & & \\ & & & & f_1^{14} & & f_2^{12} & & \end{array}$$

As one discovers on performing the iteration described below, the values of θ, ϕ, ψ arising when $k = 6$ and 7 satisfy inequality (9.2). Then proceeding as described in §2, using the iterative sequence above, the equations for λ_7, θ, ϕ and ψ are determined by

$$P \tilde{H}_2 \tilde{M}_2 M_3 Q_3^{\lambda_4^*} \approx P^{1/2} (\tilde{H}_3 \tilde{M}_3)^{2/3} Q_3^{2\lambda_6^*/3}, \tag{9.4}$$

$$P H_1 M_1 M_2 Q_2^{\lambda_5^*} \approx \Big(P (\tilde{H}_2 \tilde{M}_2)^2 M_3^8 Q_2^{\lambda_6^*} Q_3^{\lambda_4^*} \Big)^{1/2}, \tag{9.5}$$

$$P M_1 Q_1^{\lambda_6^*} \approx \Big(P \left(H_1 M_1\right)^2 M_2^{10} Q_1^{\lambda_7} Q_2^{\lambda_5^*} \Big)^{1/2}, \tag{9.6}$$

$$P^{\lambda_7} \approx P M_1^{12} Q_1^{\lambda_6^*}. \tag{9.7}$$

On writing $\delta = 4\lambda_6^* - 6\lambda_4^*$, equation (9.4) leads to the equation

$$\delta(1-\theta-\phi-\psi) + (2k-2)(\theta+\phi) - (4k+2)\psi - 3 = 0,$$

and hence

$$\psi = \frac{(2k-2-\delta)(\theta+\phi) - 3 + \delta}{4k+2+\delta}. \tag{9.8}$$

Similarly, on writing

$$\mathcal{E}_2 = \lambda_6^* - 2\lambda_5^* + \lambda_4^*, \tag{9.9}$$

equation (9.5) leads to the equation

$$\mathcal{E}_2(1-\theta-\phi) + 1 - 2k\phi = (\lambda_4^* - 8)\psi. \tag{9.10}$$

Write

$$\alpha_2 = \frac{\lambda_4^* - 8}{4k+2+\delta}.$$

Then (9.8) and (9.10) yield

$$\phi = \frac{1 + \mathcal{E}_2(1-\theta) + \alpha_2\left(3 - \delta - (2k-2-\delta)\theta\right)}{2k + \mathcal{E}_2 + \alpha_2(2k-2-\delta)}. \tag{9.11}$$

On writing

$$\mathcal{E}_1 = \lambda_7 - 2\lambda_6^* + \lambda_5^*, \tag{9.12}$$

equation (9.6) leads to the equation

$$\mathcal{E}_1(1-\theta) + 1 - 2k\theta = (\lambda_5^* - 10)\phi. \tag{9.13}$$

Write

$$\alpha_1 = \frac{\lambda_5^* - 10}{2k + \mathcal{E}_2 + \alpha_2(2k-2-\delta)}.$$

Then (9.11) and (9.13) yield

$$\theta = \frac{1 + \mathcal{E}_1 - \alpha_1\left(1 + \mathcal{E}_2 + \alpha_2(3-\delta)\right)}{2k + \mathcal{E}_1 - \alpha_1\left(\mathcal{E}_2 + \alpha_2(2k-2-\delta)\right)}.$$

From (9.7), the next iterate for λ_7 is therefore given by

$$\lambda_7' = \lambda_6^*(1-\theta) + 1 + 12\theta. \tag{9.14}$$

The values of λ_7^* obtained through the use of this iterative procedure are displayed in the Appendix.

(ii) $s = 7$ and $k = 8, 9$.

In these cases we adopt the iterative procedure displayed below.

$$\begin{array}{ccccccccc} F_0^2 f_0^{12} & \longmapsto F_1 f_1^{12} & \longrightarrow & F_2 f_2^{10} & \longrightarrow & F_3 f_3^8 & \Longrightarrow & (F_3^2)^{\frac{5}{14}}(F_3^4)^{\frac{1}{14}}(f_3^{14})^{\frac{4}{7}} \\ & & & \big\downarrow & & \big\downarrow & & \\ & & & f_1^{14} & & f_2^{12} & & \end{array}$$

The argument of part (i) of this section holds, but with (9.4) replaced by

$$P\tilde{H}_2\tilde{M}_2 M_3 Q_3^{\lambda_4^*} \approx P^{1/2}\tilde{H}_3^{4/7}\tilde{M}_3^{1-\frac{3}{7}e}Q_3^{4\lambda_7/7}.$$

On writing $\delta = 8\lambda_7 - 14\lambda_4^*$, this modified relation leads to the equation

$$\delta(1-\theta-\phi-\psi) + (6k-6e)(\theta+\phi) - (8k+6e)\psi - 11 = 0,$$

and hence

$$\psi = \frac{(6k-6e-\delta)(\theta+\phi) - 11 + \delta}{8k+6e+\delta}.$$

Write

$$\alpha_2 = \frac{\lambda_4^* - 8}{8k+6e+\delta}.$$

Then proceeding as in case (i), we find that

$$\phi = \frac{1+\mathcal{E}_2(1-\theta) + \alpha_2(11-\delta-(6k-6e-\delta)\theta)}{2k+\mathcal{E}_2+\alpha_2(6k-6e-\delta)},$$

where $\mathcal{E}_2$ satisfies (9.9). Next, on writing

$$\alpha_1 = \frac{\lambda_5^* - 10}{2k+\mathcal{E}_2+\alpha_2(6k-6e-\delta)},$$

we find that

$$\theta = \frac{1+\mathcal{E}_1 - \alpha_1(1+\mathcal{E}_2+\alpha_2(11-\delta))}{2k+\mathcal{E}_1-\alpha_1(\mathcal{E}_2+\alpha_2(6k-6e-\delta))},$$

where $\mathcal{E}_1$ satisfies (9.12). With these definitions, the next iterate λ_7' can be calculated via (9.14) once again. The converged values of λ_7^* are given in the Appendix.

(iii) $s = 8$ and $k = 6, 7, 8, 9$.

In these cases we use the following iterative procedure.

$$\begin{array}{ccccccccc} F_0^2 f_0^{14} & \longmapsto F_1 f_1^{14} & \longrightarrow & F_2 f_2^{12} & \longrightarrow & F_3 f_3^{10} & \Longrightarrow & (F_3^4)^{\frac{1}{4}}(f_3^{12})^{\frac{1}{4}}(f_3^{14})^{\frac{1}{2}} \\ & & & \big\downarrow & & \big\downarrow & & \\ & & & f_1^{16} & & f_2^{14} & & \end{array}$$

As one discovers on performing the iteration described below, the values of θ, ϕ, ψ arising when $k = 6$ and 7 satisfy inequality (9.2). Then proceeding as described in §2, using the iterative sequence above, the equations for λ_8, θ, ϕ and ψ are determined by

$$P\tilde{H}_2\tilde{M}_2M_3Q_3^{\lambda_5^*} \approx P^{1/2}\tilde{H}_3^{3/4}\tilde{M}_3^{1-\frac{e}{4}}Q_3^{\lambda_6^*/4}Q_3^{\lambda_7^*/2},$$

$$PH_1M_1M_2Q_2^{\lambda_6^*} \approx \left(P(\tilde{H}_2\tilde{M}_2)^2M_3^{10}Q_2^{\lambda_7^*}Q_3^{\lambda_5^*}\right)^{1/2},$$

$$PM_1Q_1^{\lambda_7^*} \approx \left(P(H_1M_1)^2M_2^{12}Q_1^{\lambda_8}Q_2^{\lambda_6^*}\right)^{1/2},$$

$$P^{\lambda_8} \approx PM_1^{14}Q_1^{\lambda_7^*}.$$

Let

$$\delta = 2\lambda_7^* + \lambda_6^* - 4\lambda_5^*,$$

$$\alpha_2 = \frac{\lambda_5^* - 10}{3k+e+\delta},$$

$$\mathcal{E}_2 = \lambda_7^* - 2\lambda_6^* + \lambda_5^*,$$

$$\alpha_1 = \frac{\lambda_6^* - 12}{2k + \mathcal{E}_2 + \alpha_2(k-e-\delta)},$$

$$\mathcal{E}_1 = \lambda_8 - 2\lambda_7^* + \lambda_6^*.$$

Then, arguing as in previous cases we obtain

$$\psi = \frac{(k-e-\delta)(\theta+\phi) - 1 + \delta}{3k+e+\delta},$$

$$\phi = \frac{1 + \mathcal{E}_2(1-\theta) + \alpha_2\left(1 - \delta - (k-e-\delta)\theta\right)}{2k + \mathcal{E}_2 + \alpha_2(k-e-\delta)},$$

$$\theta = \frac{1 + \mathcal{E}_1 - \alpha_1\left(1 + \mathcal{E}_2 + \alpha_2(1-\delta)\right)}{2k + \mathcal{E}_1 - \alpha_1\left(\mathcal{E}_2 + \alpha_2(k-e-\delta)\right)}.$$

The next iterate for λ_8 is given by

$$\lambda_8' = \lambda_7^*(1-\theta) + 1 + 14\theta.$$

The converged values of λ_8^* are given in the Appendix.

(iv) $s = 9$ and $k = 8$.

In this case we use the following scheme.

$$\begin{array}{ccccccccc} F_0^2 f_0^{16} & \longmapsto & F_1 f_1^{16} & \longrightarrow & F_2 f_2^{14} & \longrightarrow & F_3 f_3^{12} & \Longrightarrow & (F_3^4)^{1/4}(f_3^{16})^{3/4} \\ & & & & \downarrow & & \downarrow & & \\ & & & & f_1^{18} & & f_2^{16} & & \end{array}$$

The equations for λ_9, θ, ϕ and ψ are now determined by

$$P\tilde{H}_2\tilde{M}_2M_3Q_3^{\lambda_6^*} \approx P^{1/2}(\tilde{H}_3\tilde{M}_3)^{3/4}Q_3^{\frac{3}{4}\lambda_8^*},$$

$$PH_1M_1M_2Q_2^{\lambda_7^*} \approx \left(P(\tilde{H}_2\tilde{M}_2)^2M_3^{12}Q_2^{\lambda_8^*}Q_3^{\lambda_6^*}\right)^{1/2},$$

$$PM_1Q_1^{\lambda_8^*} \approx \left(P(H_1M_1)^2M_2^{14}Q_1^{\lambda_9}Q_2^{\lambda_7^*}\right)^{1/2},$$

$$P^{\lambda_9} \approx PM_1^{16}Q_1^{\lambda_8^*}.$$

Let

$$\delta = 3\lambda_8^* - 4\lambda_6^*,$$
$$\alpha_2 = \frac{\lambda_6^* - 12}{3k + 1 + \delta},$$
$$\mathcal{E}_2 = \lambda_8^* - 2\lambda_7^* + \lambda_6^*,$$
$$\alpha_1 = \frac{\lambda_7^* - 14}{2k + \mathcal{E}_2 + \alpha_2(k - 1 - \delta)},$$
$$\mathcal{E}_1 = \lambda_9 - 2\lambda_8^* + \lambda_7^*.$$

Then, arguing as in previous cases we obtain

$$\psi = \frac{(k-1-\delta)(\theta+\phi) - 1 + \delta}{3k + 1 + \delta},$$
$$\phi = \frac{1 + \mathcal{E}_2(1-\theta) + \alpha_2\left(1 - \delta - (k-1-\delta)\theta\right)}{2k + \mathcal{E}_2 + \alpha_2(k-1-\delta)},$$
$$\theta = \frac{1 + \mathcal{E}_1 - \alpha_1\left(1 + \mathcal{E}_2 + \alpha_2(1-\delta)\right)}{2k + \mathcal{E}_1 - \alpha_1\left(\mathcal{E}_2 + \alpha_2(k-1-\delta)\right)}.$$

The next iterate for λ_9 is given by

$$\lambda_9' = \lambda_8^*(1-\theta) + 1 + 16\theta.$$

The converged value of λ_9^* is given in the Appendix.

10. The iterative process for $k \geq 7$: fourth differences

In the analysis of the iterative procedures involving fourth differences, we follow the pattern established in previous sections. In our applications of Theorems 3.10 and 3.11, we require bounds on certain $\lambda_s^{(2Jk)}$. By using Theorem 1.4 of Vaughan [**9**], in the same manner as at the start of §8, we find that

$$\lambda_4^{(14)} \leq 4.10200120,\ \lambda_4^{(16)} \leq 4.08542333,\ \lambda_4^{(32)} \leq 4.03655147,\ \lambda_4^{(36)} \leq 4.03192910.$$

Then by Theorem 3.10(Ib) case (ii), and Theorem 3.11(I) case (ii), we have the bounds

$$\int_0^1 |F_4(\alpha)|^2 d\alpha \ll P^{1+\varepsilon}\tilde{H}_4\tilde{M}_4, \quad \int_0^1 |F_4(\alpha)|^4 d\alpha \ll P^{2+\varepsilon}\tilde{H}_4^3\tilde{M}_4^3,$$

whenever $\phi_1 \leq \Theta(k)$, where

$$\Theta(7) = 0.140805, \quad \Theta(8) = 0.124431, \quad \Theta(9) = 0.110718.$$

It transpires that for $k = 7, 8, 9$ the condition $\phi_1 \leq \Theta(k)$ is always met in the cases considered here.

(i) $s = 9$ and $k = 7$.

In this case we adopt the following iterative scheme.

$$\begin{array}{ccccccccc} F_0^2 f_0^{16} \longmapsto & F_1 f_1^{16} \longrightarrow & F_2 f_2^{14} & \longrightarrow & F_3 f_3^{12} & \longrightarrow & F_4 f_4^{10} & \Longrightarrow & (F_4^2)^{\frac14}(F_4^4)^{\frac18}(f_4^{16})^{\frac58} \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & f_1^{18} & & f_2^{16} & & f_3^{14} & & \end{array}$$

Thus λ_9, and $\phi_1, \dots, \phi_4$ are determined by the equations

$$\begin{aligned} P\tilde{H}_3\tilde{M}_3M_4Q_4^{\lambda_5^*} &\approx P^{1/2}(\tilde{H}_4\tilde{M}_4)^{5/8}Q_4^{5\lambda_8^*/8}, \\ P\tilde{H}_2\tilde{M}_2M_3Q_3^{\lambda_6^*} &\approx \Big(P(\tilde{H}_3\tilde{M}_3)^2M_4^{10}Q_3^{\lambda_7^*}Q_4^{\lambda_5^*}\Big)^{1/2}, \\ PH_1M_1M_2Q_2^{\lambda_7^*} &\approx \Big(P(\tilde{H}_2\tilde{M}_2)^2M_3^{12}Q_2^{\lambda_8^*}Q_3^{\lambda_6^*}\Big)^{1/2}, \\ PM_1Q_1^{\lambda_8^*} &\approx \Big(P(H_1M_1)^2M_2^{14}Q_1^{\lambda_9}Q_2^{\lambda_7^*}\Big)^{1/2}, \\ P^{\lambda_9} &\approx PM_1^{16}Q_1^{\lambda_8^*}. \end{aligned} \tag{10.1}$$

On writing $\delta = 5\lambda_8^* - 8\lambda_5^*$, we obtain

$$\phi_4 = \frac{(3k-3-\delta)(\phi_1+\phi_2+\phi_3) - 8 + \delta}{5k+3+\delta}.$$

Next, on writing

$$\begin{aligned} \mathcal{E}_3 &= \lambda_7^* - 2\lambda_6^* + \lambda_5^*, \\ \alpha_3 &= \frac{\lambda_5^* - 10}{5k+3+\delta}, \end{aligned} \tag{10.2}$$

we have

$$\phi_3 = \frac{1 + \mathcal{E}_3(1-\phi_1-\phi_2) + \alpha_3\left(8 - \delta - (3k-3-\delta)(\phi_1+\phi_2)\right)}{2k + \mathcal{E}_3 + \alpha_3(3k-3-\delta)}.$$

Then, on writing

$$\begin{aligned} \mathcal{E}_2 &= \lambda_8^* - 2\lambda_7^* + \lambda_6^*, \\ \alpha_2 &= \frac{\lambda_6^* - 12}{2k + \mathcal{E}_3 + \alpha_3(3k-3-\delta)}, \end{aligned} \tag{10.3}$$

we find that

$$\phi_2 = \frac{1 + (\mathcal{E}_2 - \alpha_2\mathcal{E}_3)(1-\phi_1) - \alpha_2\left(1 + \alpha_3\left(8 - \delta - (3k-3-\delta)\phi_1\right)\right)}{2k + \mathcal{E}_2 - \alpha_2\left(\mathcal{E}_3 + \alpha_3(3k-3-\delta)\right)}.$$

Finally, on writing

$$\mathcal{E}_1 = \lambda_9 - 2\lambda_8^* + \lambda_7^*, \qquad (10.4)$$
$$\alpha_1 = \frac{\lambda_7^* - 14}{2k + \mathcal{E}_2 - \alpha_2\left(\mathcal{E}_3 + \alpha_3(3k-3-\delta)\right)},$$

we deduce that

$$\phi_1 = \frac{1 + \mathcal{E}_1 - \alpha_1\left(1 + \mathcal{E}_2 - \alpha_2\left(1 + \mathcal{E}_3 + \alpha_3(8-\delta)\right)\right)}{2k + \mathcal{E}_1 - \alpha_1\left(\mathcal{E}_2 - \alpha_2\left(\mathcal{E}_3 + \alpha_3(3k-3-\delta)\right)\right)}.$$

The next iterate for λ_9 is given by

$$\lambda_9' = \lambda_8^*(1-\phi_1) + 1 + 16\phi_1. \qquad (10.5)$$

The converged value of λ_9^* is given in the Appendix.

(ii) $s = 9$ and $k = 9$.

In this case we adopt the following iterative scheme.

$$\begin{array}{ccccccccc} F_0^2 f_0^{16} & \longmapsto & F_1 f_1^{16} & \longrightarrow & F_2 f_2^{14} & \longrightarrow & F_3 f_3^{12} & \longrightarrow & F_4 f_4^{10} \Longrightarrow (F_4^2)^{\frac{7}{18}} (F_4^4)^{\frac{1}{18}} (f_4^{18})^{\frac{5}{9}} \\ & & & & \downarrow & & \downarrow & & \downarrow \\ & & & & f_1^{18} & & f_2^{16} & & f_3^{14} \end{array}$$

Thus, on replacing (10.1) by the equation

$$P\tilde{H}_3\tilde{M}_3M_4Q_4^{\lambda_5^*} \approx P^{\frac{1}{2}}(\tilde{H}_4\tilde{M}_4)^{5/9}Q_4^{5\lambda_9/9},$$

and leaving the remaining defining equations unchanged, we may apply an analogous analysis to that used in part (i) of this section. Thus we obtain

$$\phi_4 = \frac{(8k-8-\delta)(\phi_1+\phi_2+\phi_3) - 23 + \delta}{10k+8+\delta},$$

where $\delta = 10\lambda_9 - 18\lambda_5^*$. Next we obtain

$$\phi_3 = \frac{1 + \mathcal{E}_3(1-\phi_1-\phi_2) + \alpha_3\left(23 - \delta - (8k-8-\delta)(\phi_1+\phi_2)\right)}{2k + \mathcal{E}_3 + \alpha_3(8k-8-\delta)},$$

where $\mathcal{E}_3$ satisfies (10.2), and

$$\alpha_3 = \frac{\lambda_5^* - 10}{10k+8+\delta}.$$

Then we find that

$$\phi_2 = \frac{1 + (\mathcal{E}_2 - \alpha_2\mathcal{E}_3)(1-\phi_1) - \alpha_2\left(1 + \alpha_3\left(23 - \delta - (8k-8-\delta)\phi_1\right)\right)}{2k + \mathcal{E}_2 - \alpha_2\left(\mathcal{E}_3 + \alpha_3(8k-8-\delta)\right)},$$

where $\mathcal{E}_2$ satisfies (10.3), and

$$\alpha_2 = \frac{\lambda_6^* - 12}{2k + \mathcal{E}_3 + \alpha_3(8k - 8 - \delta)}.$$

Finally, we deduce that

$$\phi_1 = \frac{1 + \mathcal{E}_1 - \alpha_1 \left(1 + \mathcal{E}_2 - \alpha_2 \left(1 + \mathcal{E}_3 + \alpha_3(23 - \delta)\right)\right)}{2k + \mathcal{E}_1 - \alpha_1 \left(\mathcal{E}_2 - \alpha_2(\mathcal{E}_3 + \alpha_3(8k - 8 - \delta))\right)},$$

where $\mathcal{E}_1$ satisfies (10.4), and

$$\alpha_1 = \frac{\lambda_7^* - 14}{2k + \mathcal{E}_2 - \alpha_2 \left(\mathcal{E}_3 + \alpha_3(8k - 8 - \delta)\right)}.$$

The next iterate for λ_9 is then given by (10.5), and thus we obtain the converged value of λ_9^* given in the Appendix.

(iii) $s = 10$ and $k = 7, 8, 9$.

In these cases we use the iterative scheme displayed below.

$$\begin{array}{ccccccccccc}
F_0^2 f_0^{18} & \longmapsto & F_1 f_1^{18} & \longrightarrow & F_2 f_2^{16} & \longrightarrow & F_3 f_3^{14} & \longrightarrow & F_4 f_4^{12} & \Longrightarrow & (F_4^4)^{1/4}(f_4^{16})^{3/4} \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & f_1^{20} & & f_2^{18} & & f_3^{16} & &
\end{array}$$

Thus λ_{10} and $\phi_1, \dots, \phi_4$ are determined by the equations

$$\begin{aligned}
P\tilde{H}_3\tilde{M}_3M_4Q_4^{\lambda_6^*} &\approx P^{1/2}(\tilde{H}_4\tilde{M}_4)^{3/4}Q_4^{3\lambda_8^*/4}, \\
P\tilde{H}_2\tilde{M}_2M_3Q_3^{\lambda_7^*} &\approx \left(P(\tilde{H}_3\tilde{M}_3)^2M_4^{12}Q_3^{\lambda_8^*}Q_4^{\lambda_6^*}\right)^{1/2}, \\
PH_1M_1M_2Q_2^{\lambda_8^*} &\approx \left(P(\tilde{H}_2\tilde{M}_2)^2M_3^{14}Q_2^{\lambda_9^*}Q_3^{\lambda_7^*}\right)^{1/2}, \\
PM_1Q_1^{\lambda_9^*} &\approx \left(P(H_1M_1)^2M_2^{16}Q_1^{\lambda_{10}}Q_2^{\lambda_8^*}\right)^{1/2}, \\
P^{\lambda_{10}} &\approx PM_1^{18}Q_1^{\lambda_9^*}.
\end{aligned}$$

Hence we obtain

$$\phi_4 = \frac{(k - 1 - \delta)(\phi_1 + \phi_2 + \phi_3) - 2 + \delta}{3k + 1 + \delta} \tag{10.6}$$

where $\delta = 3\lambda_8^* - 4\lambda_6^*$. Next we find that

$$\phi_3 = \frac{1 + \mathcal{E}_3(1 - \phi_1 - \phi_2) + \alpha_3\left(2 - \delta - (k - 1 - \delta)(\phi_1 + \phi_2)\right)}{2k + \mathcal{E}_3 + \alpha_3(k - 1 - \delta)}, \tag{10.7}$$

where $\mathcal{E}_3 = \lambda_8^* - 2\lambda_7^* + \lambda_6^*$ and

$$\alpha_3 = \frac{\lambda_6^* - 12}{3k + 1 + \delta}.$$

Then

$$\phi_2 = \frac{1 + (\mathcal{E}_2 - \alpha_2 \mathcal{E}_3)(1-\phi_1) - \alpha_2\left(1 + \alpha_3\left(2 - \delta - (k-1-\delta)\phi_1\right)\right)}{2k + \mathcal{E}_2 - \alpha_2\left(\mathcal{E}_3 + \alpha_3(k-1-\delta)\right)}, \qquad (10.8)$$

where $\mathcal{E}_2 = \lambda_9^* - 2\lambda_8^* + \lambda_7^*$ and

$$\alpha_2 = \frac{\lambda_7^* - 14}{2k + \mathcal{E}_3 + \alpha_3(k-1-\delta)}.$$

Finally, we deduce that

$$\phi_1 = \frac{1 + \mathcal{E}_1 - \alpha_1\left(1 + \mathcal{E}_2 - \alpha_2\left(1 + \mathcal{E}_3 + \alpha_3(2-\delta)\right)\right)}{2k + \mathcal{E}_1 - \alpha_1\left(\mathcal{E}_2 - \alpha_2\left(\mathcal{E}_3 + \alpha_3(k-1-\delta)\right)\right)}, \qquad (10.9)$$

where $\mathcal{E}_1 = \lambda_{10} - 2\lambda_9^* + \lambda_8^*$ and

$$\alpha_1 = \frac{\lambda_8^* - 16}{2k + \mathcal{E}_2 - \alpha_2\left(\mathcal{E}_3 + \alpha_3(k-1-\delta)\right)}.$$

The next iterate for λ_{10} is then given by

$$\lambda_{10}' = \lambda_9^*(1-\phi_1) + 1 + 18\phi_1,$$

and thus we obtain the converged values of λ_{10}^* given in the Appendix.

(iv) $s = 11$ and $k = 7, 8, 9$.

In these cases we use the iterative scheme displayed below.

$$\begin{array}{ccccccccc}
F_0^2 f_0^{20} & \longmapsto & F_1 f_1^{20} & \longrightarrow & F_2 f_2^{18} & \longrightarrow & F_3 f_3^{16} & \longrightarrow & F_4 f_4^{14} \Longrightarrow (F_4^4)^{\frac14}(f_4^{18})^{\frac12}(f_4^{20})^{\frac14} \\
 & & & & \downarrow & & \downarrow & & \downarrow \\
 & & & & f_1^{22} & & f_2^{20} & & f_3^{18}
\end{array}$$

Thus λ_{11} and $\phi_1, \dots, \phi_4$ are determined by the equations

$$\begin{aligned}
P\tilde{H}_3\tilde{M}_3 M_4 Q_4^{\lambda_7^*} &\approx P^{1/2}(\tilde{H}_4\tilde{M}_4)^{3/4} Q_4^{\frac12\lambda_9^* + \frac14\lambda_{10}^*}, \\
P\tilde{H}_2\tilde{M}_2 M_3 Q_3^{\lambda_8^*} &\approx \left(P(\tilde{H}_3\tilde{M}_3)^2 M_4^{14} Q_3^{\lambda_9^*} Q_4^{\lambda_7^*}\right)^{1/2}, \\
PH_1 M_1 M_2 Q_2^{\lambda_9^*} &\approx \left(P(\tilde{H}_2\tilde{M}_2)^2 M_3^{16} Q_2^{\lambda_{10}^*} Q_3^{\lambda_8^*}\right)^{1/2}, \\
PM_1 Q_1^{\lambda_{10}^*} &\approx \left(P(H_1 M_1)^2 M_2^{18} Q_1^{\lambda_{11}} Q_2^{\lambda_9^*}\right)^{1/2}, \\
P^{\lambda_{11}} &\approx PM_1^{20} Q_1^{\lambda_{10}^*}.
\end{aligned}$$

Hence we obtain ϕ_4 as in (10.6), but with $\delta = \lambda_{10}^* + 2\lambda_9^* - 4\lambda_7^*$. Next we find that ϕ_3 is as in (10.7), but with $\mathcal{E}_3 = \lambda_9^* - 2\lambda_8^* + \lambda_7^*$ and

$$\alpha_3 = \frac{\lambda_7^* - 14}{3k + 1 + \delta}.$$

Then ϕ_2 is as in (10.8), but with $\mathcal{E}_2 = \lambda_{10}^* - 2\lambda_9^* + \lambda_8^*$ and

$$\alpha_2 = \frac{\lambda_8^* - 16}{2k + \mathcal{E}_3 + \alpha_3(k - 1 - \delta)}.$$

Finally, we deduce that ϕ_1 is as in (10.9), but with $\mathcal{E}_1 = \lambda_{11} - 2\lambda_{10}^* + \lambda_9^*$ and

$$\alpha_1 = \frac{\lambda_9^* - 18}{2k + \mathcal{E}_2 - \alpha_2(\mathcal{E}_3 + \alpha_3(k - 1 - \delta))}.$$

The next iterate for λ_{11} is then given by

$$\lambda_{11}' = \lambda_{10}^*(1 - \phi_1) + 1 + 20\phi_1,$$

and thus we obtain the converged values of λ_{11}^* given in the Appendix.

(v) $s = 12$ and $k = 7$.

In this case we use the following iterative scheme.

$$\begin{array}{ccccccccc} F_0^2 f_0^{22} & \longmapsto & F_1 f_1^{22} & \longrightarrow & F_2 f_2^{20} & \longrightarrow & F_3 f_3^{18} & \longrightarrow & F_4 f_4^{16} \Longrightarrow (F_4^4)^{\frac14}(f_4^{20})^{\frac14}(f_4^{22})^{\frac12} \\ & & & & \downarrow & & \downarrow & & \downarrow \\ & & & & f_1^{24} & & f_2^{22} & & f_3^{20} \end{array}$$

Thus λ_{12} and $\phi_1, \dots, \phi_4$ are determined by the equations

$$\begin{aligned} P\tilde{H}_3\tilde{M}_3 M_4 Q_4^{\lambda_8^*} &\approx P^{1/2}\tilde{H}_4^{3/4}\tilde{M}_4^{3/4}Q_4^{\frac14\lambda_{10}^* + \frac12\lambda_{11}^*}, \\ P\tilde{H}_2\tilde{M}_2 M_3 Q_3^{\lambda_9^*} &\approx \Big(P(\tilde{H}_3\tilde{M}_3)^2 M_4^{16} Q_3^{\lambda_{10}^*} Q_4^{\lambda_8^*}\Big)^{1/2}, \\ PH_1M_1M_2Q_2^{\lambda_{10}^*} &\approx \Big(P(\tilde{H}_2\tilde{M}_2)^2 M_3^{18} Q_2^{\lambda_{11}^*} Q_3^{\lambda_9^*}\Big)^{1/2}, \\ PM_1Q_1^{\lambda_{11}^*} &\approx \Big(P(H_1M_1)^2 M_2^{20} Q_1^{\lambda_{12}} Q_2^{\lambda_{10}^*}\Big)^{1/2}, \\ P^{\lambda_{12}} &\approx PM_1^{22}Q_1^{\lambda_{11}^*}. \end{aligned}$$

Hence we obtain ϕ_4 as in (10.6), but with $\delta = \lambda_{10}^* + 2\lambda_{11}^* - 4\lambda_8^*$. Next we find that ϕ_3 is as in (10.7), but with $\mathcal{E}_3 = \lambda_{10}^* - 2\lambda_9^* + \lambda_8^*$ and

$$\alpha_3 = \frac{\lambda_8^* - 16}{3k + 1 + \delta}.$$

Then ϕ_2 is as in (10.8), but with $\mathcal{E}_2 = \lambda_{11}^* - 2\lambda_{10}^* + \lambda_9^*$ and

$$\alpha_2 = \frac{\lambda_9^* - 18}{2k + \mathcal{E}_3 + \alpha_3(k - 1 - \delta)}.$$

Finally, we deduce that ϕ_1 is as in (10.9), but with $\mathcal{E}_1 = \lambda_{12} - 2\lambda_{11}^* + \lambda_{10}^*$ and

$$\alpha_1 = \frac{\lambda_{10}^* - 20}{2k + \mathcal{E}_2 - \alpha_2(\mathcal{E}_3 + \alpha_3(k - 1 - \delta))}.$$

The next iterate for λ_{12} is then given by

$$\lambda_{12}' = \lambda_{11}^*(1 - \phi_1) + 1 + 22\phi_1,$$

and thus we obtain the converged value of λ_{12}^* listed in the Appendix.

11. The iterative process for $k \geq 8$: Fifth and sixth differences

In the following analyses we once again follow the pattern established in previous sections. In our applications of Theorem 3.10 and 3.11, we require bounds on certain $\lambda_s^{(2Jk)}$. By using Theorem 1.4 of Vaughan [**9**] and Lemma 3.2 of Wooley [**13**], we find that

$$\lambda_3^{(16)} \leq 3.0099996, \quad \lambda_3^{(18)} \leq 3.0076932, \quad \lambda_5^{(16)} \leq 5.2248045,$$

$$\lambda_6^{(16)} \leq 6.4002032, \quad \lambda_6^{(18)} \leq 6.3497957.$$

(a) When $k = 8$, by Theorem 3.11(I), whenever

$$\phi_1 \leq 0.119329 \tag{11.1}$$

we have the estimate

$$\int_0^1 |F_5(\alpha)|^4 d\alpha \ll P^{2+\varepsilon} \tilde{H}_5^3 \tilde{M}_5^{3+\tau}, \tag{11.2}$$

where

$$\tau = \tfrac{1}{5}\left(\lambda_2^{(16)} + \lambda_3^{(16)}\right) - 1 \leq 0.002000.$$

(b) When $k = 9$, by Theorem 3.11(I) case (iii), the estimate (11.2) holds with $\tau = 0$ provided that

$$\sum_{i=1}^{I} \phi_i + k(\phi_{I-1} + \phi_I) \leq 2 \tag{11.3}$$

when $I = 3, 4, 5$. Also, by Theorem 3.11(I), whenever

$$\phi_1 \leq 0.107131 \tag{11.4}$$

we have the estimate

$$\int_0^1 |F_6(\alpha)|^4 d\alpha \ll P^{2+\varepsilon} \tilde{H}_6^3 \tilde{M}_6^{3+\tau}, \tag{11.5}$$

where

$$\tau = \tfrac{1}{3}\lambda_3^{(18)} - 1 \leq 0.002565.$$

Further, by Theorem 3.11(I) case (iii), the estimate (11.5) holds with $\tau = 0$ provided that (11.3) holds when $I = 3, 4, 5, 6$. Under the same condition, by Theorem 3.10(Ib) case (iii) we have

$$\int_0^1 |F_6(\alpha)|^2 d\alpha \ll P^{1+\varepsilon} \tilde{H}_6 \tilde{M}_6. \tag{11.6}$$

Naturally, we may use the weaker estimates contained in Theorems 3.10 and 3.11 in order to obtain a good approximation to the converged solution. In the cases under consideration, this amounts merely to using a slightly inflated value of τ.

(i) $k=8$ and $s=12,13,14,15$, and $k=9$ and $s=12,13$.

In each of these cases we use the scheme

$$\begin{array}{llllll} F_0^2 f_0^{2s-2} & & & & & \\ \downarrow & & & & & \\ F_1 f_1^{2s-2} & \to F_2 f_2^{2s-4} \to & F_3 f_3^{2s-6} \to & F_4 f_4^{2s-8} \to & F_5 f_5^{2s-10} \to & (F_5^4)^{\frac14} (f_5^{2t-2})^{a_s} (f_5^{2t})^{b_s} \\ & \quad\downarrow & \quad\downarrow & \quad\downarrow & \quad\downarrow & \\ & \quad f_1^{2s} & \quad f_2^{2s-2} & \quad f_3^{2s-4} & \quad f_4^{2s-6} & \end{array}$$

where

$$t=\left[\tfrac13(4s-17)\right], \quad a_s=\tfrac34\theta, \quad b_s=\tfrac34(1-\theta), \quad \theta=t-\tfrac13(4s-20).$$

It transpires that in the execution of the iterative process described below, the values of $\boldsymbol{\phi}$ which arise ultimately satisfy condition (11.1) when $k=8$, and condition (11.3) when $k=9$. Thus (11.2) holds with

$$\tau=\begin{cases} 0.002000 & \text{when } k=8,\\ 0 & \text{when } k=9.\end{cases}$$

Then λ_s and $\boldsymbol{\phi}$ are determined by the equations

$$P\tilde H_4 \tilde M_5 Q_5^{\lambda^*_{s-5}} \approx P^{\frac12} \tilde H_5^{\frac34} \tilde M_5^{\frac{3+\tau}{4}} Q_5^{\frac34\theta\lambda^*_{t-1}+\frac34(1-\theta)\lambda^*_t}, \tag{11.7}$$

$$P\tilde H_{j-1}\tilde M_j Q_j^{\lambda^*_{s-j}} \approx \Big(P(\tilde H_j \tilde M_j)^2 M_{j+1}^{2(s-j-1)} Q_j^{\lambda^*_{s-j+1}} Q_{j+1}^{\lambda^*_{s-j-1}}\Big)^{1/2} \quad (j=4,3,2), \tag{11.8}$$

$$PM_1 Q_1^{\lambda^*_{s-1}} \approx \Big(P(H_1M_1)^2 M_2^{2s-4} Q_1^{\lambda_s} Q_2^{\lambda^*_{s-2}}\Big)^{1/2}, \tag{11.9}$$

$$P^{\lambda_s} \approx PM_1^{2s-2} Q_1^{\lambda^*_{s-1}}. \tag{11.10}$$

Let

$$\delta = 3\theta\lambda^*_{t-1}+(3-3\theta)\lambda^*_t-4\lambda^*_{s-5},$$

$$\mathcal{E}_1=\lambda_s-2\lambda^*_{s-1}+\lambda^*_{s-2}, \tag{11.11}$$

$$\mathcal{E}_j=\lambda^*_{s-j+1}-2\lambda^*_{s-j}+\lambda^*_{s-j-1} \qquad (j=2,3,4). \tag{11.12}$$

Write

$$\kappa_j=2(s-j)-\lambda^*_{s-j} \quad (2\le j\le 5). \tag{11.13}$$

Define

$$\alpha_5=(3k+1+\delta-\tau)^{-1}, \quad \beta_5=-k+1+\delta-\tau, \quad \gamma_5=\delta-3,$$

and for $j = 4, 3, 2, 1$, define α_j, β_j and γ_j successively by

$$\gamma_j = 1 + \mathcal{E}_j + \kappa_{j+1}\alpha_{j+1}\gamma_{j+1}, \tag{11.14}$$

$$\beta_j = \mathcal{E}_j + \kappa_{j+1}\alpha_{j+1}\beta_{j+1}, \tag{11.15}$$

$$\alpha_j = (2k + \beta_j)^{-1}. \tag{11.16}$$

Then we find that $\boldsymbol{\phi}$ and λ'_s satisfy

$$\phi_j = \alpha_j\left(\gamma_j - \beta_j(\phi_1 + \cdots + \phi_{j-1})\right) \qquad (1 \le j \le 5), \tag{11.17}$$

and

$$\lambda'_s = \lambda^*_{s-1}(1 - \phi_1) + 1 + (2s - 2)\phi_1. \tag{11.18}$$

The values of λ^*_s obtained in this way are displayed in the Appendix.

(ii) $k = 9$ and $s = 14$.

In this case we use the following scheme.

$$\begin{array}{lllllll}
F_0^2 f_0^{26} & & & & & & \\
\downarrow & & & & & & \\
F_1 f_1^{26} & \to F_2 f_2^{24} \to & F_3 f_3^{22} \to & F_4 f_4^{20} \to & F_5 f_5^{18} \to & F_6 f_6^{16} \to & \left(F_6^2\right)^{\frac{1}{22}} \left(F_6^4\right)^{\frac{5}{22}} \left(f_6^{22}\right)^{\frac{8}{11}} \\
 & \quad\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
 & \quad f_1^{28} & f_2^{26} & f_3^{24} & f_4^{22} & f_5^{20} &
\end{array}$$

It transpires that in the execution of the iterative process described below, the values of $\boldsymbol{\phi}$ which arise ultimately satisfy condition (11.3) for $I = 3, 4, 5, 6$. Consequently, λ_{14} and $\boldsymbol{\phi}$ are determined by the equation

$$P\tilde{H}_5\tilde{M}_6 Q_6^{\lambda_8^*} \approx P^{\frac{1}{2}}(\tilde{H}_6\tilde{M}_6)^{\frac{8}{11}} Q_6^{\frac{8}{11}\lambda^*_{11}},$$

together with (11.8) (for $s = 14$ and $2 \le j \le 5$), and (11.9) and (11.10) (with $s = 14$). Let

$$\delta = 16\lambda^*_{11} - 22\lambda^*_8, \quad \alpha_6 = (16k + 6 + \delta)^{-1}, \quad \beta_6 = -6k + 6 + \delta, \quad \gamma_6 = \delta - 25.$$

Then with $s = 14$ and $k = 9$, we find that $\boldsymbol{\phi}$ and λ'_{14} satisfy (11.17) ($1 \le j \le 6$) and (11.18), with (11.13) ($1 \le j \le 5$), and for $j = 5, \ldots, 1$, (11.14), (11.15), (11.16).

The value of λ^*_{14} obtained in this way is given in the Appendix.

(iii) $k = 9$ and $s = 15, 16, 17, 18$.

In these cases we use the scheme

$$\begin{array}{llllll}
F_0^2 f_0^{2s-2} & & & & & \\
\downarrow & & & & & \\
F_1 f_1^{2s-2} & \to F_2 f_2^{2s-4} \to & F_3 f_3^{2s-6} \to & \ldots \to & F_6 f_6^{2s-12} \to & (F_6^4)^{\frac{1}{4}} (f_6^{2t-2})^{a_s} (f_6^{2t})^{b_s} \\
 & \quad\downarrow & \downarrow & & \downarrow & \\
 & \quad f_1^{2s} & f_2^{2s-2} & & f_5^{2s-8} &
\end{array}$$

where

$$t = \left[\tfrac{1}{3}(4s-21)\right], \quad a_s = \tfrac{3}{4}\theta, \quad b_s = \tfrac{3}{4}(1-\theta), \quad \theta = t - \tfrac{1}{3}(4s-24).$$

It transpires that in the execution of the iterative process described below, the values of $\boldsymbol{\phi}$ which arise ultimately satisfy condition (11.4). Thus, on taking $\tau = 0.002565$, we find that λ_s and $\boldsymbol{\phi}$ are determined by the equation

$$P\tilde{H}_5\tilde{M}_6 Q_6^{\lambda^*_{s-6}} \approx P^{\frac{1}{2}}\tilde{H}_6^{\frac{3}{4}}\tilde{M}_6^{\frac{3+\tau}{4}} Q_6^{\frac{3}{4}\theta\lambda^*_{t-1}+\frac{3}{4}(1-\theta)\lambda^*_t},$$

together with (11.8) (with $j = 5, 4, 3, 2$), (11.9) and (11.10). Let

$$\delta = 3\theta\lambda^*_{t-1} + (3-3\theta)\lambda^*_t - 4\lambda^*_{s-6},$$

and define $\mathcal{E}_j$ as in (11.11) and (11.12) $(2 \le j \le 5)$. Also, let

$$\alpha_6 = (3k+1+\delta-\tau)^{-1}, \quad \beta_6 = -k+1+\delta-\tau, \quad \gamma_6 = \delta - 4.$$

Then with $k = 9$, we find that the $\boldsymbol{\phi}$ satisfy (11.17) $(1 \le j \le 6)$ and (11.18), with (11.13) $(1 \le j \le 5)$, and for $1 \le j \le 5$, (11.14), (11.15), (11.16).

The values of λ^*_s obtained in this way are given in the Appendix.

12. The iterative scheme for sixth powers: $s \ge 9$.

For $s \ge 9$, our treatment of sixth powers requires a Hardy-Littlewod dissection. Further, since our conclusion entails the use of all available savings, the treatment requires considerable attention to detail. The next iterates for λ_9, λ_{10}, λ_{11} and λ_{12} are mutually dependent, and so we are forced to iterate these values collectively. Our exposition will be facilitated by first recording some preliminary lemmata.

Lemma 12.1. *Let t, u and v be positive integers exceeding 3, and let $\omega = \frac{1}{t-1}\left(\frac{1}{u}+\frac{1}{v}\right)$. Suppose that $\phi_1 \ge \phi_2 \ge \phi_3 > \frac{4}{45}$,*

$$U \le \min\left\{M_3, PH_1H_2H_3^{-3}, Q_3^{\frac{1}{4}}, Q_3M_3^{-23/6}\right\}, \tag{12.1}$$

and

$$Z = PU^{1-t\omega}\left(P^{\frac{1}{3}}M_3^{2t-12-\mu_t}\right)^{\omega}\left(P^{\frac{1}{3}}M_2^{2u-12-\mu_u}\right)^{1/u}\left(P^{\frac{1}{3}}M_3^{2v-12-\mu_v}\right)^{1/v}.$$

Then

$$\int_0^1 \left|F_3(\alpha)f_3(\alpha)^{16}\right| d\alpha \ll P^{1+\varepsilon}\tilde{M}_3\tilde{H}_3\left(Z^{-\frac{1}{4}}Q_3^{\lambda^*_8} + Q_3^{\frac{1}{4}\lambda_{10}+\frac{1}{2}\lambda_{11}-\frac{3}{2}}\right).$$

Proof. By standard Weyl differencing we have

$$|F_3(\alpha)|^2 \ll P(\tilde{M}_3\tilde{H}_3)^2 + \tilde{M}_3\tilde{H}_3|G(\alpha)|, \tag{12.2}$$

where

$$G(\alpha) = \sum_{\mathbf{h}} \sum_{\mathbf{m}} \sum_{h \le P_3} \sum_{0 < z \le P_3 - h} e\left(\alpha 2^{-6} \Psi_4(2z + h; 2\mathbf{h}, h; \mathbf{m}, 1)\right),$$

and the summations are over $\mathbf{m}$ and $\mathbf{h}$ satisfying (2.1). Write

$$\mathcal{C}(M) = \mathcal{A}(MR, R) \cap (M, MR]. \tag{12.3}$$

Recalling (3.1), (4.1) and (4.2), we may follow the analysis of the proof of Lemma 6.1 to deduce that

$$G(\alpha)^2 \ll P^{\varepsilon} D(\alpha) E_1(\alpha)^{1 - t\omega} E_2(\alpha)^{\omega} E_3(\alpha)^{1/u} E_4(\alpha)^{1/v}, \tag{12.4}$$

where

$$\begin{aligned} D(\alpha) &= D_3(\alpha; P, \boldsymbol{\phi}), \\ E_1(\alpha) &= E_1\left(\alpha; 7680 H_1 H_2 P_3, H_3, M_3; \mathcal{C}(M_3)\right), \\ E_2(\alpha) &= E_t\left(\alpha; 7680 H_1 H_2 P_3, H_3, M_3; \mathcal{C}(M_3)\right), \\ E_3(\alpha) &= E_u\left(\alpha; 3840 H_1 H_3 P_3, 2H_2, M_2; \mathcal{C}(M_2)\right), \\ E_4(\alpha) &= E_v\left(\alpha; 1920 H_2 H_3 P_3, 4H_1, M_1; \mathcal{C}(M_1)\right). \end{aligned}$$

We now recall Definition 4.9. Suppose that $\alpha \in \mathfrak{m}_3$. By Dirichlet's theorem there exist $b \in \mathbb{Z}$ and $r \in \mathbb{N}$ with

$$(b, r) = 1, \quad r \le P^{-1} Q_3^6 \quad \text{and} \quad |\alpha r - b| \le P Q_3^{-6}.$$

On noting that our assumptions on $\boldsymbol{\phi}$ imply that $P \le P^{-1} Q_3^6$, we deduce from Lemma 4.1 that

$$D(\alpha) \ll P^{\varepsilon} \left(\frac{Q_3^6}{r + Q_3^6 |\alpha r - b|} + P^{-1} Q_3^6 \right).$$

But $\alpha \in \mathfrak{m}_3$, so either $r > P$ or $Q_3^6 |\alpha r - b| \gg P R^{-18}$, and hence

$$D(\alpha) \ll P^{\varepsilon - 1} Q_3^6 \ll P^{2+\varepsilon} \tilde{H}_3. \tag{12.5}$$

Next, since $U \le M_3$, we have $U^6 \le P$, and hence

$$U^6 \le P^{-1} Q_3^6 \le U^{-6} Q_3^6.$$

Then by Lemma 4.3, we have

$$\begin{aligned} E_1(\alpha) &\ll P^{1+\varepsilon} \tilde{H}_3 M_3^2 \left(\left(r + Q_3^6 |\alpha r - b| \right)^{-1/6} + U^{-1} \right) \\ &\ll P^{1+\varepsilon} \tilde{H}_3 M_3^2 U^{-1}. \end{aligned} \tag{12.6}$$

We now observe that our hypotheses on ϕ imply that

$$8P_3H_1H_2M_3^{12}H_3^{-3} \geq M_3^{30}(M_1M_2)^{-6} \geq P^{2/3}, \quad \text{and} \quad M_3^{12} \geq P^{1/3}.$$

Then we may apply Lemma 4.6, with $Y = P^{1/3}$ and $X = P^{-1}Q_3^6$, to deduce that

$$E_2(\alpha) \ll P^{1+\varepsilon}\tilde{H}_3M_3^{\mu_t+12}\left(\left(r + Q_3^6|\alpha r - b|\right)^{-1/3} + P^{-1/3}\right)$$
$$\ll P^{2/3+\varepsilon}\tilde{H}_3M_3^{\mu_t+12}. \tag{12.7}$$

Similarly, we have

$$E_3(\alpha) \ll P^{2/3+\varepsilon}\tilde{H}_3M_2^{\mu_u+12}, \tag{12.8}$$

and

$$E_4(\alpha) \ll P^{2/3+\varepsilon}\tilde{H}_3M_1^{\mu_v+12}. \tag{12.9}$$

Thus, by (12.2), (12.4) and (12.5)-(12.9), we have

$$\sup_{\alpha\in\mathfrak{m}_3} |F_3(\alpha)| \ll P^{1+\varepsilon}\tilde{H}_3\tilde{M}_3Z^{-1/4}. \tag{12.10}$$

Now suppose that $\alpha \in \mathfrak{M}_3$. By Dirichlet's theorem there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a, q) = 1$ and satisfying (4.24). Then since $\alpha \notin \mathfrak{m}_3$, such an a and q exist with $0 \leq a \leq q \leq P$. Thus, by Lemma 4.7 we have

$$F_3(\alpha) \ll F_3^*(\alpha) + P^{\frac{2}{3}+\varepsilon}\tilde{H}_3\tilde{M}_3, \tag{12.11}$$

where $F_3^*(\alpha)$ is defined as in Definition 4.9(iii). Our hypotheses on t, u, v and ϕ imply that

$$Z^{1/4} \leq \left(PU^{1-t\omega}P^{\frac{1}{3}t\omega}\right)^{1/4} \leq P^{1/3},$$

and so by (12.10) and (12.11) we deduce that

$$\int_0^1 \left|F_3(\alpha)f_3(\alpha)^{16}\right| d\alpha \ll P^{1+\varepsilon}\tilde{H}_3\tilde{M}_3Z^{-1/4}Q_3^{\lambda_8^*} + I, \tag{12.12}$$

where

$$I = \int_{\mathfrak{M}_3} \left|F_3^*(\alpha)f_3(\alpha)^{16}\right| d\alpha.$$

But by Hölder's inequality,

$$I \ll J_1^{1/4}J_2^{1/2}J_3^{1/4}, \tag{12.13}$$

where

$$J_1 = \int_0^1 |f_3(\alpha)|^{20}\, d\alpha, \quad J_2 = \int_0^1 |f_3(\alpha)|^{22}\, d\alpha, \quad \text{and} \quad J_3 = \int_{\mathfrak{M}_3} |F_3^*(\alpha)|^4\, d\alpha.$$

We have $J_1 \ll Q_3^{\lambda_{10}+\varepsilon}$ and $J_2 \ll Q_3^{\lambda_{11}+\varepsilon}$. Further, by Lemma 4.10 we have $J_3 \ll P^\varepsilon(P\tilde{H}_3\tilde{M}_3)^4Q_3^{-6}$. The lemma now follows by (12.12) and (12.13).

Our analysis will be simplified by the use of the following lemma. We write

$$f(\alpha; Q) = \sum_{x\in\mathcal{A}(Q,R)} e(\alpha x^k).$$

Lemma 12.2. *Suppose that* $\lambda_{12} - 18 < \frac{1}{64}$. *Then*

$$\int_0^1 |f(\alpha;Q)|^{25} d\alpha \ll Q^{19+\varepsilon}.$$

Proof. Write $\Delta = \lambda_{12} - 18$. Then, by an argument mirroring precisely the proof of Theorem 1.8 of Vaughan [**8**], we may draw the following conclusion. Suppose that $0 < \delta < \frac{1}{12}$. Let $\mathfrak{m}$ denote the set of real numbers α with the property that whenever $a \in \mathbb{Z}$, $q \in \mathbb{N}$, $(a,q) = 1$ and $|\alpha - a/q| \le q^{-1} Q^{\frac{1}{2}+6\delta-6}$, then one has $q > Q^{\frac{1}{2}+6\delta}$, and let $\rho = \frac{1}{48}(1-\Delta)$. Then

$$\sup_{\alpha\in\mathfrak{m}} |f(\alpha;Q)| \ll Q^{1+\varepsilon}(Q^{-\delta} + Q^{-\rho}).$$

We take $\delta = \frac{1}{64}$. Then by hypothesis we have $\rho > \frac{1}{64}$, and hence

$$\int_{\mathfrak{m}} |f(\alpha;Q)|^{25} d\alpha \ll Q^{\frac{63}{64}+\varepsilon} \int_0^1 |f(\alpha;Q)|^{24} d\alpha \ll Q^{19+\varepsilon}. \tag{12.14}$$

Now suppose that $\alpha \notin \mathfrak{m}$. Then by Dirichlet's theorem we may choose a and q with

$$(a,q) = 1, \quad |q\alpha - a| \le Q^{\frac{1}{2}+6\delta-6} \quad \text{and} \quad q \le Q^{\frac{1}{2}+6\delta}. \tag{12.15}$$

We write $\mathfrak{M}(q,a)$ for the set of such α satisfying (12.15), and $\mathfrak{M}$ for the union of the $\mathfrak{M}(q,a)$ with $(a,q) = 1$ and $1 \le a \le q \le Q^{\frac{1}{2}+6\delta}$. Then if $\alpha \in \mathfrak{M}(q,a)$, by Lemma 7.2 of Vaughan and Wooley [**10**] we have

$$f(\alpha;Q) \ll Q^{1+\varepsilon}\left(\left(q + Q^6|\alpha q - a|\right)^{-\frac{1}{12}} + Q^{-\frac{1}{64}}\right). \tag{12.16}$$

Define $V^*(\alpha)$ to be the function of α taking the value zero whenever $\alpha \in \mathfrak{m}$, and by

$$V^*(\alpha) = Q^{1+\varepsilon}\left(q + Q^6|\alpha q - a|\right)^{-\frac{1}{12}}$$

whenever $\alpha \in \mathfrak{M}(q,a)$ with $(a,q) = 1$ and $0 \le a \le q \le P$. Then from (12.14) and (12.16) it follows that

$$\int_0^1 |f(\alpha;Q)|^{25} d\alpha \ll \int_0^1 V^*(\alpha)|f(\alpha;Q)|^{24} d\alpha + Q^{19+\varepsilon}.$$

But by Hölder's inequality, the latter integral is

$$\ll \left(\int_0^1 V^*(\alpha)^{25} d\alpha\right)^{\frac{1}{25}} \left(\int_0^1 |f(\alpha;Q)|^{25} d\alpha\right)^{\frac{24}{25}}.$$

Thus

$$\int_0^1 |f(\alpha;Q)|^{25} d\alpha \ll Q^{19+\varepsilon} + \int_0^1 V^*(\alpha)^{25} d\alpha \ll Q^{19+\varepsilon},$$

which completes the proof of the lemma.

Lemma 12.3. *Let t and u be positive integers exceeding* 3*, and $\omega = \frac{1}{(t-1)u}$. Let v be either* 9 *or* 10*, and define $K(9) = \frac{3}{5}\lambda_{11} + \frac{1}{5}\lambda_{12} - \frac{6}{5}$, and*

$$K(10) = \begin{cases} \frac{2}{5}\lambda_{12} + \frac{2}{5}\lambda_{13} - \frac{6}{5} & \text{when } \lambda_{12} - 18 \geq \frac{1}{64}, \\ \\ \quad 14 & \text{when } \lambda_{12} - 18 < \frac{1}{64}. \end{cases}$$

Suppose that $\phi_1 \geq \phi_2 \geq \frac{1}{18}$, and

$$Z = PM_1^{1-t\omega}\left(P^{1/3}M_1^{2t-12-\mu_t}\right)^{\omega}\left(P^{1/3}M_2^{2u-12-\mu_u}\right)^{1/u}.$$

Then

$$\int_0^1 |F_2(\alpha)f_2(\alpha)^{2v}|d\alpha \ll P^{1+\varepsilon}\tilde{M}_2\tilde{H}_2\left(Z^{-\frac{1}{8}}Q_2^{\lambda_v} + Q_2^{K(v)}\right).$$

Proof. By standard Weyl differencing we have

$$|F_2(\alpha)|^4 \leq P^3(\tilde{M}_2\tilde{H}_2)^4 + P(\tilde{M}_2\tilde{H}_2)^3|G(\alpha)|, \tag{12.17}$$

where

$$G(\alpha) = \sum_{\mathbf{h}}\sum_{\mathbf{m}}\sum_{l_1\leq P_2}\sum_{l_2\leq P_2}\sum_{0<z\leq P_2-l_1-l_2} e\left(2^{-6}\alpha\Psi_4(2z+l_1+l_2;2\mathbf{h},l_1,l_2;\mathbf{m},1,1)\right),$$

and the summations are over $\mathbf{m}$ and $\mathbf{h}$ satisfying (2.1). Recalling (12.3), (3.1), (4.1) and (4.2), we may follow the analysis of the proof of Lemma 6.1 to deduce that

$$G(\alpha)^2 \ll P^{\varepsilon}D(\alpha)E_1(\alpha)^{1-t\omega}E_2(\alpha)^{\omega}E_3(\alpha)^{1/u}, \tag{12.18}$$

where

$$\begin{aligned} D(\alpha) &= D_2(\alpha;P,\boldsymbol{\phi}), \\ E_1(\alpha) &= E_1(\alpha;480H_2P_2^2,2H_1,M_1;\mathcal{C}(M_1)), \\ E_2(\alpha) &= E_t(\alpha;480H_2P_2^2,2H_1,M_1;\mathcal{C}(M_1)), \\ E_3(\alpha) &= E_u(\alpha;960H_1P_2^2,H_2,M_2;\mathcal{C}(M_2)). \end{aligned}$$

We now recall Definition 4.9. Suppose that $\alpha \in \mathfrak{m}_2$. By Dirichlet's theorem, there exist $b \in \mathbb{Z}$ and $r \in \mathbb{N}$ with

$$(b,r) = 1, \quad r \leq P^{-1}Q_2^6 \quad \text{and} \quad |\alpha r - b| \leq PQ_2^{-6}.$$

Therefore, by Lemma 4.1, we have

$$D(\alpha) \ll P^{\varepsilon}\left(\frac{Q_2^6}{r + Q_2^6|\alpha r - b|} + P^{-1}Q_2^6\right).$$

But $\alpha \in \mathfrak{m}_2$, so either $r > P$ or $Q_2^6|\alpha r - b| \gg PR^{-24}$, and hence

$$D(\alpha) \ll P^{\varepsilon-1}Q_2^6 \ll P^{3+\varepsilon}\tilde{H}_2. \tag{12.19}$$

Next, on noting that

$$M_1^6 \le P^{-1}Q_2^6 \le M_1^{-6}Q_2^6,$$

we may apply Corollary 4.2.1 with $X = P^{-1}Q_2^6$ to deduce that

$$\begin{aligned} E_1(\alpha) &\ll P^{2+\varepsilon}\tilde{H}_2M_1^2\left(\left(r + Q_2^6|\alpha r - b|\right)^{-1/6} + M_1^{-1}\right) \\ &\ll P^{2+\varepsilon}H_1H_2M_1. \end{aligned} \tag{12.20}$$

We now observe that our hypotheses on $\boldsymbol{\phi}$ imply that

$$2P_2^2H_1M_2^{12}H_2^{-3} \ge M_2^{30}M_1^{-6} \ge P^{2/3}, \quad \text{and} \quad M_2^{12} \ge P^{1/3}.$$

Then we may apply Lemma 4.6, with $Y = P^{1/3}$ and $X = P^{-1}Q_2^6$, to deduce that

$$\begin{aligned} E_3(\alpha) &\ll P^{2+\varepsilon}\tilde{H}_2M_2^{\mu_t+12}\left(\left(r + Q_2^6|\alpha r - b|\right)^{-1/3} + P^{-1/3}\right) \\ &\ll P^{5/3+\varepsilon}\tilde{H}_2M_2^{\mu_t+12}. \end{aligned} \tag{12.21}$$

Similarly, we have

$$E_2(\alpha) \ll P^{5/3+\varepsilon}\tilde{H}_2M_1^{\mu_u+12}. \tag{12.22}$$

Thus, by (12.17)-(12.22), we have

$$\sup_{\alpha\in\mathfrak{m}_2}|F_2(\alpha)| \ll P^{1+\varepsilon}\tilde{H}_2\tilde{M}_2Z^{-1/8}. \tag{12.23}$$

Now suppose that $\alpha \in \mathfrak{M}_2$. By Dirichlet's theorem there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a, q) = 1$ and satisfying (4.24). Then since $\alpha \notin \mathfrak{m}_2$, such an a and q exist with $0 \le a \le q \le P$. Thus, by Lemma 4.7 we have

$$F_2(\alpha) \ll F_2^*(\alpha) + P^{\frac{3}{4}+\varepsilon}\tilde{H}_2\tilde{M}_2, \tag{12.24}$$

where $F_2^*(\alpha)$ is as in Definition 4.9(iii). Our hypotheses on ϕ imply that

$$Z \le P^{5/3}M_1 \le P^2,$$

and so by (12.23) and (12.24) we deduce that

$$\int_0^1 |F_2(\alpha)f_2(\alpha)^{2v}|d\alpha \ll P^{1+\varepsilon}\tilde{H}_2\tilde{M}_2Z^{-1/8}Q_2^{\lambda_v} + I, \tag{12.25}$$

where

$$I = \int_{\mathfrak{M}_2}\left|F_2^*(\alpha)f_2(\alpha)^{2v}\right|d\alpha.$$

But by Hölder's inequality,

$$I \ll J_1^{4/5} J_2^{1/5}, \tag{12.26}$$

where

$$J_1 = \int_0^1 |f_2(\alpha)|^{\frac{5}{2}v} d\alpha \quad \text{and} \quad J_2 = \int_{\mathfrak{M}_2} |F_2^*(\alpha)|^5 d\alpha.$$

By Lemma 4.10 we have

$$J_2 \ll P^\varepsilon (P\tilde{H}_2\tilde{M}_2)^5 Q_2^{-6}. \tag{12.27}$$

Also, by Hölder's inequality, when $v = 9$ we have

$$J_1 \ll \left(Q_2^{\lambda_{11}+\varepsilon}\right)^{\frac{3}{4}} \left(Q_2^{\lambda_{12}+\varepsilon}\right)^{\frac{1}{4}}, \tag{12.28}$$

and when $v = 10$ we have

$$J_1 \ll \left(Q_2^{\lambda_{12}+\varepsilon}\right)^{\frac{1}{2}} \left(Q_2^{\lambda_{13}+\varepsilon}\right)^{\frac{1}{2}}. \tag{12.29}$$

Further, when $\lambda_{12} - 18 < \frac{1}{64}$, we apply Lemma 12.2 and obtain

$$\int_0^1 |f_2(\alpha)|^{25} d\alpha \ll Q_2^{19+\varepsilon}. \tag{12.30}$$

The lemma now follows on combining (12.25)-(12.30).

We shall find, in future analyses, that it is convenient to have a modified form of Lemma 12.3.

Lemma 12.4. *Let t be a positive integer exceeding* 3*, and $\mathcal{B} \subseteq (1, P]$. Define*

$$H_{\mathcal{B}}(\alpha) = \sum_{\mathbf{h}} \sum_{\substack{M_1 < m_1 \le M_1 R \\ m_1 \in \mathcal{B}}} \sum_{\substack{M_2 < m_2 \le M_2 R \\ m_2 \in \mathcal{A}(P,R)}} \sum_z e\left(\alpha \Psi_2(z, \mathbf{h}, \mathbf{m})\right),$$

where the summation is with $\mathbf{h}$ satisfying (2.1)*. Suppose that $\phi_1 \ge \phi_2 \ge \frac{1}{18}$, and*

$$Z = PM_1^{1-\frac{1}{t}} \left(P^{\frac{1}{3}} M_2^{2t-12-\mu_t}\right)^{1/t}.$$

Then

$$\int_0^1 \left|H_{\mathcal{B}}(\alpha) f_2(\alpha)^{20}\right| d\alpha \ll P^{1+\varepsilon} \tilde{H}_2 \tilde{M}_2 \left(Z^{-\frac{1}{8}} Q_2^{\lambda_{10}} + Q_2^{K(10)}\right),$$

where $K(10)$ is defined in the statement of Lemma 12.3.

Proof. By standard Weyl differencing we have

$$|H_{\mathcal{B}}(\alpha)|^4 \le P^3 (\tilde{H}_2\tilde{M}_2)^4 + P(\tilde{H}_2\tilde{M}_2)^3 |G(\alpha)|,$$

with $G(\alpha)$ defined as in the proof of Lemma 12.3 save with the variable m_1 ranging over $m_1 \in \mathcal{B}$. Write

$$\mathcal{C}_1 = \{m \in \mathcal{B} : M_1 < m \le M_1 R\} \quad \text{and} \quad \mathcal{C}_2 = \{m \in \mathcal{A}(P,R) : M_2 < m \le M_2 R\}.$$

Then recalling (3.1), (4.1) and (4.2), we may follow the proof of Lemma 6.1 to deduce that

$$G(\alpha)^2 \ll P^{\varepsilon} D(\alpha) E_1^*(\alpha)^{1-1/t} E_2^*(\alpha)^{1/t} M_1^{2/t},$$

where

$$\begin{aligned} D(\alpha) &= D_2(\alpha; P, \boldsymbol{\phi}) \\ E_1^*(\alpha) &= E_1(\alpha; 480 H_2 P_2^2, 2H_1, M_1; \mathcal{C}_1), \\ E_2^*(\alpha) &= E_t(\alpha; 960 H_1 P_2^2, H_2, M_2; \mathcal{C}_2). \end{aligned}$$

The proof now continues in precisely the same manner as that of Lemma 12.3.

We now divide into cases according to the value of s. As usual we let (λ_s) be an iterate of the sequence converging to (λ_s^*), and to simplify formulae we write $\theta = \phi_1$, $\phi = \phi_2$, and $\psi = \phi_3$. We require suitable values for μ_s for various values of s. These may be obtained through the use of Lemma 3.2 of Wooley [**13**]. We record here for future reference the permissible values

$$\mu_{26} = 40.3153894, \quad \mu_{27} = 42.2641797 \quad \text{and} \quad \mu_{28} = 44.2211063.$$

(i) $s = 9$.

In this case we use the following scheme.

$$\begin{array}{ccccccccc} F_0^2 f_0^{16} & \longrightarrow & F_1 f_1^{16} & \longrightarrow & F_2 f_2^{16} & \longrightarrow & F_3 f_3^{16} & \Longrightarrow & (F_3)(f_3^{16}). \\ & & & & \downarrow & & \downarrow & & \\ & & & & f_1^{16} & & f_2^{16} & & \end{array}$$

In executing the iterative process described below, it transpires that $\boldsymbol{\phi}$ satisfies the conditions of Lemma 12.1, and moreover good choices for t, u and v are $t = 26$ and $u = v = 27$. Therefore, by Lemma 12.1 we have

$$\int_0^1 \left| F_3(\alpha) f_3(\alpha)^{16} \right| d\alpha \ll P^{\varepsilon}(\mathcal{U}_1 + \mathcal{U}_2),$$

where

$$\mathcal{U}_1 = P\tilde{H}_3 \tilde{M}_3 Z^{-1/4} Q_3^{\lambda_8^*},$$

and

$$\mathcal{U}_2 = P\tilde{H}_3 \tilde{M}_3 Q_3^{\frac{1}{4}\lambda_{10} + \frac{1}{2}\lambda_{11} - \frac{3}{2}}.$$

We take Z to be as large as is consistent with the conditions of Lemma 12.1. Write $\delta = \frac{1}{4}\lambda_{10} + \frac{1}{2}\lambda_{11} - \lambda_8^* - \frac{3}{2}$. Then proceeding as described in §2, using the iterative scheme above, the equations for λ_9, θ, ϕ and ψ are determined by

$$P^{\lambda_9} \approx PM_1^{16}Q_1^{\lambda_8^*}, \tag{12.31}$$

$$PM_1Q_1^{\lambda_8^*} \approx \left(P(M_1H_1)^2M_2^{16}Q_1^{\lambda_8^*}Q_2^{\lambda_8^*}\right)^{\frac{1}{2}}, \tag{12.32}$$

$$PM_1M_2H_1Q_2^{\lambda_8^*} \approx \left(P(\tilde{M}_2\tilde{H}_2)^2M_3^{16}Q_2^{\lambda_8^*}Q_3^{\lambda_8^*}\right)^{\frac{1}{2}}, \tag{12.33}$$

$$P\tilde{M}_2M_3\tilde{H}_2Q_3^{\lambda_8^*} \approx P\tilde{M}_3\tilde{H}_3Q_3^{\lambda_8^*}\left(Z^{-\frac{1}{4}} + Q_3^{\delta}\right). \tag{12.34}$$

In our iterative process, we solve the equations (12.32)-(12.34) for ϕ subject to the constraint (12.1), and taking care to consider the contributions of both $\mathcal{U}_1$ and $\mathcal{U}_2$. The core of the method will be apparent from the explanation below, where we pay attention to the situation towards the end of the iteration process. We write $\Delta = \lambda_8^* - 10$.

For the moment, suppose that our ultimate choices for θ, ϕ and ψ imply that $\mathcal{U}_1$ is the dominating contribution. Write $\delta_1 = \mu_{26} - 40$ and $\delta_2 = \mu_{27} - 42$. Suppose, as is ultimately the case in our iteration, that

$$\tfrac{6}{17}(\theta + \phi) < \psi < \tfrac{6}{35}(1 - \theta - \phi).$$

It follows that (12.1) holds with $U = M_3$. Then the equations (12.32)-(12.34) yield

$$\begin{aligned} 4 - 24\psi &= 1 + \frac{623}{675}\psi + \frac{2(1-3\delta_1\psi)}{2025} + \frac{1-3\delta_2\theta}{81} + \frac{1-3\delta_2\phi}{81}, \\ 12\phi &= 1 + (6-\Delta)\psi, \qquad (12.35) \\ 12\theta &= 1 + (6-\Delta)\phi. \qquad (12.36) \end{aligned}$$

Therefore

$$\psi = \frac{6023 + 75\delta_2(\theta+\phi)}{50469 - 6\delta_1},$$

and hence

$$\phi = \frac{86607 - 6\delta_1 - 6023\Delta + 75(6-\Delta)\delta_2\theta}{605628 - 72\delta_1 - 75(6-\Delta)\delta_2},$$

and

$$\theta = \frac{1125270 - 108\delta_1 - 122745\Delta + 6023\Delta^2 + 6\delta_1\Delta - 75(6-\Delta)\delta_2}{7267536 - 864\delta_1 - 75(6-\Delta)(18-\Delta)\delta_2}.$$

On the other hand, if $\mathcal{U}_2$ is the dominating contribution, then equations (12.32)-(12.34) yield (12.35), (12.36), and

$$1 - 6\psi + \delta(1 - \theta - \phi - \psi) = 0.$$

Then, on writing

$$\alpha=\frac{1+\delta}{6+\delta},\quad \beta=\frac{\delta}{6+\delta}\quad\text{and}\quad \gamma=6-\Delta,$$

we obtain

$$\theta=\frac{12+\gamma(1+\beta+\gamma\alpha)}{144+\gamma\beta(12+\gamma)},\tag{12.37}$$

and

$$\phi=\frac{1+\gamma(\alpha-\beta\theta)}{12+\gamma\beta}.$$

It transpires that $\mathcal{U}_2$ is the dominating contribution. By (12.31), the next iterate for λ_9 is given by

$$\lambda_9'=\lambda_8^*(1-\theta)+1+16\theta,$$

where θ satisfies (12.37).

(ii) $s=10$ and 11.

In each of these cases we use the following scheme.

$$\begin{array}{ccccccc} F_0^2f_0^{2s-2} & \longrightarrow & F_1f_1^{2s-2} & \longrightarrow & F_2f_2^{2s-2} & \Longrightarrow & (F_2)(f_2^{2s-2}) \\ & & & & \downarrow & & \\ & & & & f_1^{2s-2} & & \end{array}$$

In executing the iterative process described below, it transpires that $\boldsymbol{\phi}$ satisfies the conditions of Lemma 12.3, and moreover good choices for t and u are $t=u=28$. We divide into cases.

(a) $s=10$.

By Lemma 12.3 we have

$$\int_0^1\left|F_2(\alpha)f_2(\alpha)^{18}\right|d\alpha\ll P^\varepsilon(\mathcal{U}_1+\mathcal{U}_2)$$

where

$$\mathcal{U}_1=P\tilde{H}_2\tilde{M}_2Z^{-\frac18}Q_2^{\lambda_9},$$

and

$$\mathcal{U}_2=P\tilde{H}_2\tilde{M}_2Q_2^{\frac35\lambda_{11}+\frac15\lambda_{12}-\frac65}.$$

Write $\delta=\frac35\lambda_{11}+\frac15\lambda_{12}-\lambda_9-\frac65$. Then, proceeding as described in §2 with the above iterative sequence for $s=10$, the equations for λ_{10}, θ and ϕ are determined by

$$P^{\lambda_{10}}\approx PM_1^{18}Q_1^{\lambda_9},\tag{12.38}$$

$$PM_1Q_1^{\lambda_9}\approx\left(P(H_1M_1)^2M_2^{18}Q_1^{\lambda_9}Q_2^{\lambda_9}\right)^{\frac12},\tag{12.39}$$

$$PH_1\tilde{M}_2Q_2^{\lambda_9}\approx P\tilde{H}_2\tilde{M}_2Q_2^{\lambda_9}\left(Z^{-1/8}+Q_2^{\delta}\right).\tag{12.40}$$

We write $\Delta = \lambda_9 - 12$.

For the time being, suppose that our ultimate choices for θ and ϕ imply that $\mathcal{U}_1$ is the dominating contribution. Write $\delta' = \mu_{28} - 44$. Then the equations (12.39) and (12.40) yield

$$\begin{aligned} 8 - 48\phi &= 1 + \frac{26}{27}\theta + \frac{1 - 3\delta'\theta}{2268} + \frac{1 - 3\delta'\phi}{84}, \\ 12\theta &= 1 + (6 - \Delta)\phi. \end{aligned} \tag{12.41}$$

Therefore

$$\phi = \frac{15848 - (2184 - 3\delta')\theta}{108864 - 81\delta'},$$

and hence

$$\theta = \frac{203952 - 15848\Delta - 81\delta'}{1319472 - 990\delta' - 2184\Delta + 3\delta'\Delta}. \tag{12.42}$$

On the other hand, if $\mathcal{U}_2$ is the dominating contribution, then equations (12.39) and (12.40) yield (12.41) and

$$1 - 6\phi + \delta(1 - \theta - \phi) = 0.$$

Write

$$\alpha = \frac{6 - \Delta}{6 + \delta}.$$

Then

$$\phi = \frac{1 + \delta(1 - \theta)}{6 + \delta}$$

and

$$\theta = \frac{1 + \alpha(1 + \delta)}{12 + \alpha\delta}. \tag{12.43}$$

As the iteration process converges, it transpires that $\mathcal{U}_1$ provides the dominating contribution. Then by (12.38), the next iterate for λ_{10} is given by

$$\lambda'_{10} = \lambda_9(1 - \theta) + 1 + 18\theta,$$

with θ given by (12.42).

(b) $s = 11$

By Lemma 12.3 we have

$$\int_0^1 \left| F_2(\alpha) f_2(\alpha)^{20} \right| d\alpha \ll P^{\varepsilon}(\mathcal{U}_1 + \mathcal{U}_2),$$

where

$$\begin{aligned} \mathcal{U}_1 &= P\tilde{H}_2\tilde{M}_2 Z^{-1/8} Q_2^{\lambda_{10}}, \\ \mathcal{U}_2 &= P\tilde{H}_2\tilde{M}_2 Q_2^{K(10)}, \end{aligned}$$

and $K(10)$ is defined as in the statement of Lemma 12.3. Write $\delta = K(10) - \lambda_{10}$. Then, proceeding as described in §2 with the above iterative sequence for $s = 11$, the equations for λ_{11}, θ and ϕ are determined by

$$P^{\lambda_{11}} \approx PM_1^{20}Q_1^{\lambda_{10}}, \tag{12.44}$$

$$PM_1Q_1^{\lambda_{10}} \approx \left(P(H_1M_1)^2M_2^{20}Q_1^{\lambda_{10}}Q_2^{\lambda_{10}}\right)^{1/2}, \tag{12.45}$$

$$PH_1M_1M_2Q_2^{\lambda_{10}} \approx P\tilde{H}_2\tilde{M}_2Q_2^{\lambda_{10}}\left(Z^{-1/8} + Q_2^{\delta}\right). \tag{12.46}$$

For the time being, suppose that our ultimate choices for θ and ϕ imply that $\mathcal{U}_1$ is the dominating contribution. Then following the pattern set in the case $s = 10$, we obtain

$$\theta = \frac{203952 - 15848\Delta - 81\delta'}{1319472 - 990\delta' - 2184\Delta + 3\delta'\Delta}$$

where $\delta' = \mu_{28} - 44$ and $\Delta = \lambda_{10} - 14$.

On the other hand, if $\mathcal{U}_2$ is the dominating contribution, then the equations (12.45) and (12.46) yield (12.41) and

$$1 - 6\phi + \delta(1 - \theta - \phi) = 0.$$

Thus, with the notation used for $s = 11$, we find that θ is given by (12.43). In order to make use of these equations, we require a suitable upper bound for λ_{13}. It suffices to use inequality $(k-2)$ of §4 of Vaughan [**8**], which gives

$$\lambda_{13} \le \max\left\{\lambda_{12}\left(1 - \frac{15}{97}\right) + 1 + 24\left(\frac{15}{97}\right), 20\right\}.$$

As the iteration process converges, it transpires that $\mathcal{U}_2$ provides the dominating contribution, and further that $L(10) = 14$ is permissible. Under such circumstances, by (12.44) the next iterate for λ_{11} is given by

$$\lambda_{11}' = \lambda_{10}(1 - \theta) + 1 + 20\theta,$$

where θ satisfies

$$\theta = \frac{2 - \Delta}{12 - \Delta}.$$

(iii) $s = 12$.

In this case we use the following scheme.

$$\begin{array}{ccccccc} F_0^2 f_0^{22} & \longrightarrow & F_1 f_1^{22} & \longrightarrow & F_2 f_2^{20} & \Longrightarrow & (F_2)(f_2^{20}). \\ & & & & \downarrow & & \\ & & & & f_1^{24} & & \end{array}$$

In executing the iterative process described below, it transpires that ϕ satisfies the conditions of Lemma 12.4, and moreover a good choice for t is $t = 28$. By Lemma 12.4 we have

$$\int_0^1 \left|F_2(\alpha) f_2(\alpha)^{20}\right| d\alpha = \int_0^1 \left|H_{\mathcal{A}(P,R)}(\alpha) f_2(\alpha)^{20}\right| d\alpha \\ \ll P^{\varepsilon}(\mathcal{U}_1 + \mathcal{U}_2),$$

where

$$\mathcal{U}_1 = P\tilde{H}_2\tilde{M}_2 Z^{-\frac{1}{8}} Q_2^{\lambda_{10}},$$

and

$$\mathcal{U}_2 = P\tilde{H}_2\tilde{M}_2 Q_2^{K(10)}.$$

Write $\delta = K(10) - \lambda_{10}$. We now proceed as for the case $s = 11$. The equations for λ_{12}, θ and ϕ are given by

$$P^{\lambda_{12}} \approx PM_1^{22} Q_1^{\lambda_{11}}, \tag{12.47}$$

$$PM_1 Q_1^{\lambda_{11}} \approx \left(P(M_1H_1)^2 M_2^{20} Q_1^{\lambda_{12}} Q_2^{\lambda_{10}}\right)^{1/2}, \tag{12.48}$$

$$PH_1M_1M_2Q_2^{\lambda_{10}} \approx P\tilde{H}_2\tilde{M}_2 Q_2^{\lambda_{10}} \left(Z^{-1/8} + Q_2^{\delta}\right). \tag{12.49}$$

For each s, define Δ_s by

$$\lambda_s = 2s - 6 + \Delta_s.$$

Let

$$\mathcal{E} = \lambda_{12} - 2\lambda_{11} + \lambda_{10}. \tag{12.50}$$

Then equations (12.48) and (12.49) yield

$$8 - 48\phi = 1 + \frac{27}{28}\theta + \frac{1 - 3\delta\phi}{84},$$
$$12\theta = 1 + \mathcal{E}(1 - \theta) + (6 - \Delta_{10})\phi. \tag{12.51}$$

Therefore

$$\phi = \frac{587 - 81\theta}{4032 - 3\delta},$$

and hence

$$\theta = \frac{7554 - 3\delta - 587\Delta_{10} + \mathcal{E}(4032 - 3\delta)}{48870 - 81\Delta_{10} - 36\delta + \mathcal{E}(4032 - 3\delta)}.$$

On the other hand, when $\mathcal{U}_2$ is the dominating contribution, the equations (12.48) and (12.49) yield (12.51) and

$$1 - 6\phi + \delta(1 - \theta - \phi) = 0. \tag{12.52}$$

Then, on writing

$$\alpha = \frac{6-\Delta_{10}}{6+\delta},$$

we find that

$$\theta = \frac{1+\mathcal{E}+\alpha(1+\delta)}{12+\mathcal{E}+\alpha\delta}. \tag{12.53}$$

As the iteration process converges we find that $\mathcal{U}_2$ is the dominating contribution. In such circumstances, by (12.47) the next iterate for λ_{12} is given by

$$\lambda'_{12} = \lambda_{11}(1-\theta)+1+22\theta, \tag{12.54}$$

where θ is given by (12.53). Moreover, as the iteration process converges, we find that $K(10) = 14$ is permissible. Thus $\delta = -\Delta_{10}$, $\alpha = 1$, and so by (12.52) and (12.53), we have

$$\phi = \frac{1-\Delta_{10}(1-\theta)}{6-\Delta_{10}},$$

and

$$\theta = \frac{2+\mathcal{E}-\Delta_{10}}{12+\mathcal{E}-\Delta_{10}}.$$

But by (12.50), we have $\mathcal{E} = \Delta_{12} - 2\Delta_{11} + \Delta_{10}$, and hence

$$\theta = \frac{2+\Delta_{12}-2\Delta_{11}}{12+\Delta_{12}-2\Delta_{11}}. \tag{12.55}$$

But by (12.54),

$$\theta = \frac{\lambda'_{12}-\lambda_{11}-1}{22-\lambda_{11}} = \frac{1+\Delta'_{12}-\Delta_{11}}{6-\Delta_{11}}, \tag{12.56}$$

by using the natural induced notation. Therefore, by equating (12.55) and (12.56), we deduce that the limit of the iteration process for λ_{11} and λ_{12} satisfies

$$\frac{1+\Delta^*_{12}-\Delta^*_{11}}{6-\Delta^*_{11}} = \frac{2+\Delta^*_{12}-2\Delta^*_{11}}{12+\Delta^*_{12}-2\Delta^*_{11}}.$$

On simplifying this expression, we obtain the equation

$$\Delta^*_{12}\,(\Delta^*_{12}-2\Delta^*_{11}+7) = 0.$$

Then since Δ^*_{12} must be non-negative, it follows that $\Delta^*_{12} = 0$, and hence $\lambda^*_{12} = 18$.

We summarise the values of λ_s arising from our method in the Appendix.

We now complete the proof of Theorem 1.1 for $k = 6$. Since $\lambda^*_{12} + 1 - \frac{1}{32} < 25$, we may conclude by the methods of §5 of Vaughan [**8**] that $G(6) \le 25$. Moreover, as is evident, we fail to obtain $G(6) \le 24$ by "ε". This is a problem to which we return in Vaughan and Wooley [**11**].

13. The Hardy-Littlewood dissection for larger k

We now return to the pattern established in the sections preceding §12. Before considering the iterative procedures themselves, we record a lemma. We shall merely sketch the proof of this lemma, the details closely resembling those of the proof of Lemma 6.1. We shall find it convenient here, and in future sections, to define the quantity Δ_s by

$$\lambda_s = 2s - k + \Delta_s.$$

Lemma 13.1. *Suppose that $j \leq k-4$. Let u be a positive integer, and define*

$$\tau = 2^{1+j-k}, \quad t = \left[\left(\frac{k-j+1}{k-j}\right)u + 1\right], \quad \theta = t - \left(\frac{k-j+1}{k-j}\right)u,$$

and

$$\nu_u = \frac{k-j}{k-j+1}\left(\theta\Delta_{t-1} + (1-\theta)\Delta_t\right).$$

Then

$$\int_0^1 |F_j(\alpha)f_j(\alpha)^{2u}|d\alpha \ll P^{1+\varepsilon}\tilde{H}_j\tilde{M}_jQ_j^{2u-k}\left((PM_1)^{-\tau}Q_j^{\Delta_u} + Q_j^{\nu_u}\right).$$

Proof. On recalling Definition 4.9, we may imitate the analysis of the proof of Lemma 6.1 to deduce that

$$\int_0^1 |F_j(\alpha)f_j(\alpha)^{2u}|d\alpha \ll I_1 + I_2, \tag{13.1}$$

where

$$I_1 = \int_0^1 \left|F_j^*(\alpha)f_j(\alpha)^{2u}\right| d\alpha,$$

and

$$I_2 = \left(P^{\frac{k-j-1}{k-j}+\varepsilon}\tilde{H}_j\tilde{M}_j + \sup_{\alpha\in\mathfrak{m}_j}|F_j(\alpha)|\right)\int_0^1 |f_j(\alpha)|^{2u}d\alpha. \tag{13.2}$$

By Hölder's inequality,

$$I_1^{k-j+1} \ll \left(\int_0^1 |F_j^*(\alpha)|^{k-j+1}d\alpha\right)\mathcal{I}_{t-1}^{(k-j)\theta}\mathcal{I}_t^{(k-j)(1-\theta)},$$

where

$$\mathcal{I}_s = \int_0^1 |f_j(\alpha)|^{2s}d\alpha \quad (s = t-1, t).$$

Then by Lemma 4.10,

$$I_1 \ll P^{1+\varepsilon}\tilde{H}_j\tilde{M}_jQ_j^{2u-k+\nu_u}. \tag{13.3}$$

Also, using a Weyl differencing argument, we may follow the pattern established in Lemmata 6.1 and 12.1 to deduce, from Lemma 4.1 and Corollary 4.2.1, that

$$\sup_{\alpha\in\mathfrak{m}_j} |F_j(\alpha)| \ll P^{1+\varepsilon}\tilde{H}_j\tilde{M}_j(PM_1)^{-\tau}. \tag{13.4}$$

The proof of the lemma is completed on combining (13.1)-(13.4).

Our iterative procedures will be based on schemes of the following form.

$$\begin{array}{ccccccccc} F_0^2 f_0^{2s-2} & \longmapsto & F_1 f_1^{2s-2} & \longrightarrow & F_2 f_2^{2s-2} & \longrightarrow \dots \longrightarrow & F_j f_j^{2s-2} & \Longrightarrow & (F_j)(f_j^{2s-2}) \\ & & & & \downarrow & & \downarrow & & \\ & & & & f_1^{2s-2} & & f_{j-1}^{2s-2} & & \end{array}$$

In order to set the scene, we start by investigating the consequences of the assumption

$$(\Delta_{s-1}-\nu_{s-1})(1-\phi_1-\cdots-\phi_j) \geq \tau(1+\phi_1), \tag{13.5}$$

where ν_{s-1} is defined as in the statement of Lemma 13.1. Since $0 \leq \phi_i \leq \frac{1}{k}$ $(1 \leq i \leq j)$, it follows that (13.5) holds whenever

$$\Delta_{s-1}-\nu_{s-1} \geq 2^{1+j-k}\frac{k+1}{k-j}. \tag{13.6}$$

By Lemma 13.1, whenever (13.5) holds, λ_s and $\boldsymbol{\phi}$ are determined by the equations

$$P\tilde{H}_{j-1}\tilde{M}_jQ_j^{\lambda_{s-1}} \approx P\tilde{H}_j\tilde{M}_jQ_j^{\lambda_{s-1}}(PM_1)^{-\tau}, \tag{13.7}$$

$$P\tilde{H}_{i-1}\tilde{M}_iQ_i^{\lambda_{s-1}} \approx \left(P(\tilde{M}_i\tilde{H}_i)^2M_{i+1}^{2s-2}Q_{i+1}^{\lambda_{s-1}}Q_i^{\lambda_{s-1}}\right)^{1/2} \quad (1 \leq i < j), \tag{13.8}$$

$$P^{\lambda_s} \approx PM_1^{2s-2}Q_1^{\lambda_{s-1}}. \tag{13.9}$$

Write $\Delta = \Delta_{s-1}$. Then equations (13.7) and (13.8) lead to the equations

$$k\phi_j = 1-\tau(1+\phi_1), \tag{13.10}$$

$$2k\phi_i = 1+(k-\Delta)\phi_{i+1} \quad (1 \leq i < j). \tag{13.11}$$

The recurrence relations (13.11) may be solved, as in Lemma 3.2 of Wooley [**13**], to give

$$\phi_i = \frac{1}{k+\Delta}+\left(\phi_j-\frac{1}{k+\Delta}\right)\left(\frac{k-\Delta}{2k}\right)^{j-i} \quad (1 \leq i < j). \tag{13.12}$$

Write

$$\alpha = \frac{k-\Delta}{2k}.$$

Then by (13.10), we have

$$\phi_1 = \frac{\frac{1}{k+\Delta} + \left(\frac{1-\tau}{k} - \frac{1}{k+\Delta}\right)\alpha^{j-1}}{1 + \frac{\tau}{k}\alpha^{j-1}}. \tag{13.13}$$

By (13.9), we find that λ_s^* is then given by

$$\lambda_s^* = \lambda_{s-1}^*(1-\phi_1) + 1 + (2s-2)\phi_1. \tag{13.14}$$

In order to check that (13.5) holds, we need to estimate ν_{s-1}. By inequality $(k-2)$ of §4 of Vaughan [**8**] (which, incidentally, is case $j=1$ of (13.13)), it follows that we may assume that for each t,

$$\Delta_{t+1} \le \max\left\{\frac{(k-1+2^{3-k})\Delta_t - (k+1)2^{2-k}}{k+2^{2-k}}\ ,\ 0\right\} \tag{13.15}$$

whence a suitable estimate for ν_{s-1} follows. Alternatively, we may apply Lemma 3.2 of Wooley [**13**], obtaining

$$\Delta_{t+1} \le \Delta_t(1-\theta) + k\theta - 1, \tag{13.16}$$

where

$$\theta = \frac{1}{k+\Delta_t} + \left(\frac{1}{k} - \frac{1}{k+\Delta_t}\right)\left(\frac{k-\Delta_t}{2k}\right)^{k-1}.$$

14. The proof of Theorem 1.1 for seventh powers.

We divide into cases according to the value of s.

(a) $s = 13$.

We use Lemma 13.1 with $j=3$. By reference to the Appendix with $s=12$, we obtain by successive application of (13.15) the bound

$$\Delta_{12} - \nu_{12} = \Delta_{12} - \tfrac{4}{5}\Delta_{15} > 0.2169,$$

and hence condition (13.6) is met. Then ϕ_1 is given by (13.13) with $\Delta = \Delta_{12}$, and we obtain the value of λ_{13}^* given in the Appendix by using (13.14).

(b) $s = 14$.

We use Lemma 13.1 with $j=2$. By successive application of (13.15) we obtain

$$\Delta_{13} - \nu_{13} = \Delta_{13} - \tfrac{1}{3}\Delta_{15} - \tfrac{1}{2}\Delta_{16} > 0.1506,$$

and hence condition (13.6) is met. Then ϕ_1 is given by (13.13) with $\Delta = \Delta_{13}$, and we obtain the value of λ_{14}^* given in the Appendix by using (13.14).

(c) $s = 15$.

We use Lemma 13.1 with $j = 2$. By successive application of (13.15) we obtain

$$\Delta_{14} - \nu_{14} = \Delta_{14} - \tfrac{1}{6}\Delta_{16} - \tfrac{2}{3}\Delta_{17} > 0.1130,$$

and hence condition (13.6) is met. Then ϕ_1 is given by (13.13) with $\Delta = \Delta_{14}$, and we obtain the value of λ_{15}^* given in the Appendix by using (13.14).

(d) $s = 16$.

In this case we are forced to modify our argument by using the following scheme.

$$\begin{array}{ccccccc} F_0^2 f_0^{30} & \longmapsto & F_1 f_1^{30} & \longrightarrow & F_2 f_2^{28} & \Longrightarrow & (F_2)(f_2^{28}) \\ & & & & \big\downarrow & & \\ & & & & f_1^{32} & & \end{array}$$

We may apply Lemma 13.1 to estimate the final integral implicit in this scheme. Thus, as in the case $s = 15$ we find that (13.6) is satisfied, and hence λ_{16}, ϕ_1 and ϕ_2 are determined by the equations (13.7) with $s = 15$ and $j = 2$,

$$PM_1Q_1^{\lambda_{15}^*} \approx \Big(P(M_1H_1)^2M_2^{28}Q_2^{\lambda_{14}^*}Q_1^{\lambda_{16}}\Big)^{1/2},$$

and (13.9) with $s = 16$. Write $\Delta = \Delta_{14}$ and $\mathcal{E} = \lambda_{16} - 2\lambda_{15}^* + \lambda_{14}^*$. Then the equations for λ_{16}, ϕ_1 and ϕ_2 are determined by the equations (13.10) and

$$2k\phi_1 = 1 + \mathcal{E}(1 - \phi_1) + (k - \Delta)\phi_2.$$

Thus

$$\phi_1 = \frac{217 + 112\mathcal{E} - 15\Delta}{1575 + 112\mathcal{E} - \Delta}.$$

By (13.9), we find that the next iterate for λ_{16} is given by

$$\lambda_{16}' = \lambda_{15}^*(1 - \phi_1) + 1 + 30\phi_1.$$

The converged value of λ_{16}^* is given in the Appendix.

Let $X = P^{\frac{k}{2k-1}}$ and $Z = PX^{-1}$. Define the generating function $h(\alpha)$ by

$$h(\alpha) = \sum_{x \in \mathcal{C}} e(\alpha x^k), \tag{14.1}$$

where

$$\mathcal{C} = \{x : x = pz, X/2 < p \leq X, z \in \mathcal{A}(Z, Z^\eta)\}.$$

Let s be an even integer, and write $s = 2r$. Define $\mathfrak{m}$ to be the set of real numbers α in $((2k)^{-1}P^{1-k}, 1 + (2k)^{-1}P^{1-k}]$ with the property that whenever $a \in \mathbb{Z}$, $q \in \mathbb{N}$, $(a, q) = 1$ and $|\alpha - a/q| \leq q^{-1}X^{1-k}(rZ^k)^{-1}$, then one has $q > X$. Then the argument of §9 of Vaughan [**8**] gives

$$\sup_{\alpha \in \mathfrak{m}} |h(\alpha)| \ll P^{1-\sigma+\varepsilon}, \tag{14.2}$$

where

$$\sigma = \frac{k - (k-1)\Delta_s}{2s(2k-1)}. \tag{14.3}$$

By (14.2) with $s = 12$, and using the value of λ_{12} given in the Appendix, we have $\sigma > 0.01679703$. Moreover, $\lambda_{16}^* + 1 - \sigma < 26$. Then by Theorem 4 of Vaughan and Wooley [**10**], we may finally conclude that $G(7) \leq 33$.

15. The proof of Theorem 1.1 for eighth powers

We divide into cases according to the value of s.

(a) $s = 16$.

We use Lemma 13.1 with $j = 3$. By reference to the Appendix with $s = 15$, we obtain by successive application of (13.15) the bound

$$\Delta_{15} - \nu_{15} = \Delta_{15} - \tfrac{5}{6}\Delta_{18} > 0.1563,$$

and hence condition (13.6) is met. Then ϕ_1 is given by (13.13) with $\Delta = \Delta_{15}$, and we obtain the value of λ_{16}^* given in the Appendix using (13.14).

(b) $s = 17$.

We use Lemma 13.1 with $j = 3$. By reference to the Appendix with $s = 16$, we obtain by successive application of (13.15) the bound

$$\Delta_{16} - \nu_{16} = \Delta_{16} - \tfrac{2}{3}\Delta_{19} - \tfrac{1}{6}\Delta_{20} > 0.1288,$$

and hence condition (13.6) is met. Then ϕ_1 is given by (13.13) with $\Delta = \Delta_{16}$, and we obtain the value of λ_{17}^* given in the Appendix using (13.14).

(c) $s = 18$.

We use Lemma 13.1 with $j = 2$. By reference to the Appendix with $s = 17$, we obtain by successive application of (13.15) the bound

$$\Delta_{17} - \nu_{17} = \Delta_{17} - \tfrac{1}{7}\Delta_{19} - \tfrac{5}{7}\Delta_{20} > 0.0937,$$

and hence condition (13.6) is met. Then ϕ_1 is given by (13.13) with $\Delta = \Delta_{17}$, and we obtain the value of λ_{18}^* given in the Appendix using (13.14).

We now complete the proof of Theorem 1.1 for $k = 8$ as in §14. Applying (14.3) with $s = 16$, we obtain $\sigma > 0.01381643$. Moreover, $\lambda_{18}^* + 7(1 - \sigma) < 35$. Then by Theorem 4 of Vaughan and Wooley [**10**], we may finally conclude that $G(8) \leq 43$.

16. The proof of Theorem 1.1 for ninth powers

We divide into cases according to the value of s.

(a) $s = 19$.

We use Lemma 13.1 with $j = 4$. By reference to the Appendix with $s = 18$, we obtain by successive application of (13.16) the bound

$$\Delta_{18} - \nu_{18} = \Delta_{18} - \tfrac{1}{3}\Delta_{21} - \tfrac{1}{2}\Delta_{22} > 0.1659,$$

and hence condition (13.6) is met. Then ϕ_1 is given by (13.13) with $\Delta = \Delta_{18}$, and we may obtain the value of λ_{19}^* given in the Appendix using (13.14).

(b) $s = 20$.

We use Lemma 13.1 with $j = 4$. By reference to the Appendix with $s = 19$, we obtain by successive application of (13.16) the bound

$$\Delta_{19} - \nu_{19} = \Delta_{19} - \tfrac{1}{6}\Delta_{22} - \tfrac{2}{3}\Delta_{23} > 0.1307,$$

and hence condition (13.6) is met. Then ϕ_1 is given by (13.13) with $\Delta = \Delta_{19}$, and we may obtain the value of λ_{20}^* given in the Appendix using (13.14).

(c) $s = 21$.

We use Lemma 13.1 with $j = 3$. By reference to the Appendix with $s = 20$, we obtain by successive application of (13.16) the bound

$$\Delta_{20} - \nu_{20} = \Delta_{20} - \tfrac{4}{7}\Delta_{23} - \tfrac{2}{7}\Delta_{24} > 0.0912,$$

and hence condition (13.6) is met. Then ϕ_1 is given by (13.13) with $\Delta = \Delta_{20}$, and we may obtain the value of λ_{21}^* given in the Appendix using (13.14).

(d) $s = 22$.

We use Lemma 13.1 with $j = 3$. By reference to the Appendix with $s = 21$, we obtain by successive application of (13.16) the bound

$$\Delta_{21} - \nu_{21} = \Delta_{21} - \tfrac{3}{7}\Delta_{24} - \tfrac{3}{7}\Delta_{25} > 0.0703,$$

and hence condition (13.6) is met. Then ϕ_1 is given by (13.13) with $\Delta = \Delta_{21}$, and we may obtain the value of λ_{22}^* given in the Appendix using (13.14).

(e) $s = 23$.

We use Lemma 13.1 with $j = 3$. By reference to the Appendix with $s = 22$, we obtain by successive application of (13.16) the bound

$$\Delta_{22} - \nu_{22} = \Delta_{22} - \tfrac{2}{7}\Delta_{25} - \tfrac{4}{7}\Delta_{26} > 0.0527,$$

and hence condition (13.6) is met. Then ϕ_1 is given by (13.13) with $\Delta = \Delta_{22}$, and we may obtain the value of λ_{23}^* given in the Appendix using (13.14).

We now complete the proof of Theorem 1.1 for $k = 9$ as in §14. Applying (14.3) with $s = 20$, we obtain $\sigma > 0.01150790$. Moreover, $\lambda_{23}^* + 5(1 - \sigma) < 42$. Then by Theorem 4 of Vaughan and Wooley [**10**], we may finally conclude that $G(9) \le 51$.

Appendix. Numerical values for parameters

In this appendix we display in tabular form the numerical values of the parameters arising in our iterative processes. The displayed figures are the converged values, calculated to 15 significant figures on a computer, and rounded up in the last digit displayed. We also give the numerical values of the $\sigma(k)$ arising from (14.3), rounded down in the last digit displayed.

$k = 5$.

s	λ_s	ϕ_1	ϕ_2
3	3.1362571	0.06812854	
4	4.4386563	0.10559577	
5	5.9250797	0.13658426	0.07226662
6	7.5417546	0.15133422	0.11310401
7	9.2727289	0.16396009	0.14346470
8	11.0773627	0.17021105	0.14377599

We note also that

$$S_9(P,R) \ll P^{13}.$$

Further, although worse than the corresponding estimate arising from Weyl's inequality, we have

$$\sigma(5) \geq 0.03257326.$$

$k = 6$.

s	λ_s	ϕ_1	ϕ_2	ϕ_3
3	3.0909091	0.04545455		
4	4.3333334	0.08333334		
5	5.7246965	0.10673541	0.05080042	
6	7.2315633	0.11855692	0.08751084	
7	8.8505716	0.12981369	0.10763684	0.05551767
8	10.5604127	0.13784851	0.12076716	0.08562337
9	12.3536709	0.14583058	0.13787203	0.12031506
10	14.2030055	0.15042244	0.14258278	
11	16.0860412	0.15232648	0.14281844	
12	18.0000000	0.15454265	0.14289604	

We note also that $S_{12}(P, R) \ll P^{18+\varepsilon}$, and although worse than Weyl's inequality, $\sigma(6) \geq 0.02301567$.

$k = 7$.

s	λ_s	ϕ_1	ϕ_2	ϕ_3	ϕ_4
3	3.0639191	0.03195955			
4	4.2641175	0.06818559			
5	5.5891167	0.08699398	0.03541170		
6	7.0143820	0.09641272	0.06937556		
7	8.5410894	0.10564538	0.08803450	0.04058919	
8	10.1526323	0.11202654	0.09889245	0.06902202	
9	11.8469485	0.11873997	0.10797294	0.08946112	0.04150797
10	13.6055676	0.12329153	0.11453127	0.09898491	0.06609542
11	15.4242973	0.12803790	0.12028445	0.10870656	0.08585428
12	17.2932208	0.13214156	0.12611292	0.11717668	0.10266360
13	19.1987053	0.13501034	0.13272313		
14	21.1230182	0.13590250	0.13271516		
15	23.0625298	0.13661685	0.13270878		
16	25.0164264	0.13749920	0.13270091		

We have $\sigma(7) \geq 0.01679703$, which is superior to Weyl's inequality.

$k = 8$.

s	λ_s	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5
3	3.0496111	0.02480553				
4	4.2289285	0.06077755				
5	5.5116307	0.07496603	0.03518923			
6	6.8806000	0.08220565	0.06261215			
7	8.3284883	0.08748844	0.07303331	0.02548707		
8	9.8579814	0.09336014	0.08199712	0.05500300		
9	11.4648635	0.09880825	0.09013287	0.07343430		
10	13.1382531	0.10304140	0.09589809	0.08328930	0.05353266	
11	14.8742074	0.10725466	0.10102623	0.09178812	0.07307451	
12	16.6623509	0.11060434	0.10540355	0.09753530	0.08383402	0.05196286
13	18.4948992	0.11346253	0.10890495	0.10229102	0.09095506	0.06779771
14	20.3659701	0.11606386	0.11215719	0.10668910	0.09798016	0.08143693
15	22.2689476	0.11828320	0.11517483	0.11083494	0.10444347	0.09360961
16	24.1954446	0.11984099	0.11867153	0.11625125		
17	26.1370265	0.12064517	0.11920252	0.11624496		
18	28.0945483	0.12177604	0.12061807			

We have $\sigma(8) \geq 0.01381643$, which is superior to Weyl's inequality.

$k = 9$.

s	λ_s	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6
3	3.0358052	0.0179026					
4	4.1822894	0.0494179					
5	5.4201075	0.0622934	0.0120224				
6	6.7434120	0.0705922	0.0500843				
7	8.1447208	0.0763440	0.0659715	0.0299044			
8	9.6154494	0.0803939	0.0729087	0.0537809			
9	11.1526889	0.0841468	0.0774905	0.0640395	0.0201554		
10	12.7545442	0.0878966	0.0819482	0.0717050	0.0452838		
11	14.4174241	0.0914891	0.0863641	0.0785379	0.0622043		
12	16.1349528	0.0946287	0.0902411	0.0839157	0.0725554	0.0447782	
13	17.9006237	0.0973520	0.0934860	0.0881279	0.0791991	0.0601626	
14	19.7094207	0.0998592	0.0964579	0.0919939	0.0851563	0.0722296	0.0413991
15	21.5537941	0.1018474	0.0989335	0.0948926	0.0889089	0.0782184	0.0547876
16	23.4293887	0.1036673	0.1010817	0.0976837	0.0925898	0.0840732	0.0669177
17	25.3311019	0.1052100	0.1030531	0.1001422	0.0960512	0.0894131	0.0771596
18	27.2542905	0.1064944	0.1047428	0.1023855	0.0990896	0.0941699	0.0856692
19	29.1946817	0.1075260	0.1069631	0.1058045	0.1034200		
20	31.1468279	0.1081331	0.1074800	0.1061450	0.1034158		
21	33.1102975	0.1088277	0.1083112	0.1072611			
22	35.0806499	0.1091547	0.1085283	0.1072599			
23	37.0566117	0.1094208	0.1087045	0.1072590			

We have $\sigma(9) \geq 0.01150790$, which is superior to Weyl's inequality.

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