WELL-DISTRIBUTION MODULO ONE AND THE PRIMES

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Abstract. Let (p_n) denote the sequence of prime numbers, with $2 = p_1 < p_2 < \ldots$. We demonstrate the existence of an irrational number α having the property that the sequence (αp_n) is not well-distributed modulo 1.

1. Introduction

Consider a real sequence (s_n) and the associated fractional parts $\{s_n\} = s_n - \lfloor s_n \rfloor$. This sequence is said to be *equidistributed* (or *uniformly distributed*) modulo 1 when, for each pair a and b of real numbers with $0 \leqslant a < b \leqslant 1$, one has

$$\lim_{N\to\infty} \frac{\operatorname{card}\{n \in [1,N] \cap \mathbb{Z} : a \leqslant \{s_n\} \leqslant b\}}{N} = b - a.$$

A stronger notion than equidistribution is obtained by insisting that, for each natural number m, the sequence (s_{n+m}) should be equidistributed modulo 1, uniformly in m. This property of being *well-distributed* modulo 1 was introduced by Petersen [5] in 1956. More concretely, we say that the sequence (s_n) is well-distributed modulo 1 when, for each pair a and b with $0 \leqslant a < b \leqslant 1$, one has

$$\lim_{N\to\infty} \sup_{m\in\mathbb{N}} \left| \frac{\operatorname{card}\{n \in [1,N] \cap \mathbb{Z} : a \leqslant \{s_{n+m}\} \leqslant b\}}{N} - (b-a) \right| = 0.$$

It is a consequence of pioneering work of Weyl [9] that, given a polynomial

$$\psi_d(t;\boldsymbol{\alpha}) = \alpha_d t^d + \ldots + \alpha_1 t + \alpha_0 \in \mathbb{R}[t]$$

with an irrational coefficient α_l for some $l \geqslant 1$, the sequence $(\psi_d(n;\boldsymbol{\alpha}))$ is equidistributed modulo 1. Moreover, Lawton [3, Theorem 2] established that, under the same conditions, this sequence satisfies the stronger property of being well-distributed modulo 1. We refer the reader to Bergelson and Moreira [2, §3] for further discussion on sequences well-distributed modulo 1.

In this note, we focus on the sequence (p_n) of prime numbers with $2 = p_1 < p_2 < \ldots$. It was famously proved by Vinogradov that when α is irrational, then the sequence (αp_n) is equidistributed modulo 1 (see [8], for example). Subsequently, a relatively simple proof of this conclusion was presented by Vaughan [7]. It is natural to enquire whether this equidistribution extends to a corresponding well-distribution property. The purpose of this note is to answer this question in the negative.

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Theorem 1.1. *There exists an irrational number α having the property that the sequence (αp_n) is not well-distributed modulo* 1.

At first sight, this conclusion may seem surprising, since the primes are undeniably equidistributed at large scales. However, as first discerned by Maier [4], the primes exhibit irregularities in their distribution at very small scales. When it comes to generating a failure of well-distribution, the most convenient manifestation of such irregularities of which to avail oneself is that established in work of Shiu [6, Theorem 1]. The latter author shows, in particular, that for each natural number q, there exist arbitrarily long strings of prime numbers $p_{n+1}, \ldots, p_{n+k}$ with $p_{n+1} \equiv \ldots \equiv p_{n+k} \equiv 1 \pmod{q}$. By carefully constructing an associated irrational number α, this congruential bias amongst consecutive primes may be shown to generate a corresponding failure of well-distribution modulo 1 in the sequence (αp_n). It will be evident from our proof of Theorem 1.1, which we present in §2, that many such numbers α can be constructed, each of which is transcendental.

As is usual, we write $e(z)$ for $e^{2\pi \mathrm{i} z}$, and for $\theta \in \mathbb{R}$ we define $\|\theta\| = \min\{|\theta - t| : t \in \mathbb{Z}\}$.

2. The application of Shiu's theorem

Our strategy for proving Theorem 1.1 depends on the careful selection of a sequence (n_k) of natural numbers with $1 \leqslant n_0 < n_1 < \ldots$, and the associated real number

$$\alpha = \sum_{k=0}^{\infty} 2^{-n_k}. \tag{2.1}$$

The sequence (n_k) is defined iteratively in terms of a consequence of Shiu's theorem on strings of congruent primes. Thus, for each $n \in \mathbb{N}$, there exists an integer $m = m(n)$ with

$$p_{m+1} \equiv \ldots \equiv p_{m+n} \equiv 1 \pmod{2^n}. \tag{2.2}$$

The conclusion of [6, Theorem 1(i)] shows that, when n is sufficiently large, such an integer $m(n)$ exists with $m(n) < \exp_4(n)$, where $\exp_r(x)$ denotes the r-fold iterated exponential function. We define the sequence (n_k) as follows. We put $n_0 = 1$, and then define

$$m_k = m(n_k), \quad \pi_k = p_{m_k+n_k}, \quad n_{k+1} = 4\pi_k \quad (k \geqslant 0).$$

We investigate the well-distribution of the sequence (αp_n) by means of an analogue of Weyl's criterion for equidistribution. Thus, as a consequence of [5, Theorems 2 and 3], we see that (αp_n) is well-distributed modulo 1 if and only if, for each $h \in \mathbb{N}$, one has

$$\lim_{N\to\infty} \sup_{m\in\mathbb{N}} \left| N^{-1} \sum_{n=1}^{N} e(h\alpha p_{n+m}) \right| = 0. \tag{2.3}$$

Before embarking on the proof of Theorem 1.1 in earnest, we pause to confirm that the real number α defined in (2.1) is irrational.

Lemma 2.1. *The number α is transcendental, and hence is irrational.*

Proof. For each $k \in \mathbb{N}$, put

$$q_k = 2^{n_k} \quad \text{and} \quad a_k = 2^{n_k} \sum_{l=0}^{k} 2^{-n_l}.$$

Then we see that $a_k \in \mathbb{N}$ and $(q_k, a_k) = 1$. Moreover, one has

$$|q_k\alpha - a_k| \leqslant \sum_{l>k} 2^{n_k - l_k}.$$

It is evident from the definition of the function $m(n)$ via (2.2) that $\pi_k > 2^{n_k}$, whence $n_{k+1} > 2^{n_k}$. Consequently, whenever k is large enough and $l > k$, one has

$$0 < \sum_{l>k} 2^{n_k - n_l} < \sum_{l>k} 2^{-l-kn_k} < q_k^{-k}.$$

We therefore deduce that $|q_k\alpha - a_k| < q_k^{-k}$. By Liouville's theorem (see [1, Theorem 1.1], for example), it therefore follows that α cannot be algebraic. Thus, we conclude that α is transcendental, and hence irrational. □

We next show that, for each positive integer h, the real number $\|h\alpha(p_n - 1)\|$ is infinitely often very small for long strings of consecutive primes p_n.

Lemma 2.2. *Let h be a positive integer. Then for each sufficiently large positive integer k, one has*

$$\|h\alpha(p_{i+m_k} - 1)\| < \pi_k^{-2} \quad (1 \leqslant i \leqslant n_k).$$

Proof. Given a positive integer h, when k is sufficiently large and $1 \leqslant i \leqslant n_k$, one has

$$0 < h(p_{i+m_k} - 1) \sum_{l>k} 2^{-n_l} \leqslant \sum_{l>k} h\pi_k 2^{-l-3\pi_k} < \pi_k^{-2}.$$

Meanwhile, under the same conditions, one finds that since $p_{i+m_k} \equiv 1 \pmod{2^{n_k}}$, then

$$h(p_{i+m_k} - 1) \sum_{l=0}^{k} 2^{-n_l} \equiv 0 \pmod 1.$$

By combining these conclusions, therefore, we infer that for $1 \leqslant i \leqslant n_k$, one has

$$\left\| h(p_{i+m_k} - 1)\left(\sum_{l=0}^{k} 2^{-n_l} + \sum_{l>k} 2^{-n_l} \right) \right\| < \pi_k^{-2},$$

and the desired conclusion follows from (2.1). □

We are now equipped to complete the proof of Theorem 1.1. Let h be a natural number. From Lemma 2.2, we find that whenever k is sufficiently large and $1 \leqslant i \leqslant n_k$, one has

$$|e(h\alpha p_{i+m_k}) - e(h\alpha)| = |e(h\alpha(p_{i+m_k} - 1)) - 1| < \pi_k^{-1}.$$

Thus, when N is an integer with $1 \leqslant N \leqslant n_k$, then

$$\left| N^{-1} \sum_{n=1}^{N} e(h\alpha p_{n+m_k}) - N^{-1} \sum_{n=1}^{N} e(h\alpha) \right| < \pi_k^{-1},$$

whence

$$1 - \pi_k^{-1} < \left| N^{-1} \sum_{n=1}^{N} e(h\alpha p_{n+m_k}) \right| \leqslant 1.$$

In particular, for each positive integer N, we infer that

$$1-\frac{1}{N}\leqslant \sup_{m\in\mathbb{N}}\left|N^{-1}\sum_{n=1}^{N}e(h\alpha p_{n+m})\right|\leqslant 1.$$

This relation confirms that

$$\lim_{N\to\infty}\sup_{m\in\mathbb{N}}\left|N^{-1}\sum_{n=1}^{N}e(h\alpha p_{n+m})\right|=1, \tag{2.4}$$

in contradiction with Weyl's criterion for well-distribution modulo 1 given in (2.3). We are therefore forced to conclude that (αp_n) is not well-distributed modulo 1. In view of Lemma 2.1, this completes the proof of Theorem 1.1.

Although the case $h=1$ of (2.4) suffices to prove Theorem 1.1, we gave a more general argument since it might be useful. We note also that the number α may be modified extensively without impairing the validity of our proof. Indeed, given an integer $q\geqslant 2$ and a sequence of positive integers (b_k) not growing too rapidly, the number α defined in (2.1) could be replaced by

$$\beta=\sum_{k=0}^{\infty}b_kq^{-n_k},$$

and still the sequence (βp_n) is not well-distributed modulo 1. Furthermore, the rapid growth of the integer n_k may be considerably weakened without damaging the crude bound of Lemma 2.2, and so the Liouville-type properties of α may also be relaxed.

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