ESTIMATES FOR SMOOTH WEYL SUMS ON MINOR ARCS

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Abstract. We provide new estimates for smooth Weyl sums on minor arcs and explore their consequences for the distribution of the fractional parts of αn^k. In particular, when $k \geqslant 6$ and $\rho(k)$ is defined via the relation $\rho(k)^{-1} = k(\log k + 8.02113)$, then for all large numbers N there is an integer n with $1 \leqslant n \leqslant N$ for which $\|\alpha n^k\| \leqslant N^{-\rho(k)}$.

1. Introduction

Estimates for smooth Weyl sums on minor arcs play a prominent role in applications of the Hardy-Littlewood method, in the study of the distribution of fractional parts of polynomial sequences, and in many other branches of the theory of numbers. When $2 \leqslant R \leqslant P$, let $\mathscr{A}(P,R)$ denote the set of natural numbers not exceeding P having all of their prime factors bounded above by R. Given a natural number $k \geqslant 2$, define the Weyl sum

$$f(\alpha;P,R) = \sum_{n\in\mathscr{A}(P,R)} e(\alpha n^k), \tag{1.1}$$

where, as usual, we write $e(z)$ for $\mathrm{e}^{2\pi \mathrm{i} z}$. In this context, a typical choice of minor arcs is the set $\mathfrak{n}$ of all real numbers α with the property that when $q \in \mathbb{N}$ and $a \in \mathbb{Z}$ are coprime with $|q\alpha - a| \leqslant P^{1-k}$, one has $q > P$. Improving earlier estimates of Vaughan [8], Wooley [13] showed that for each $\varepsilon > 0$ there is a positive number $\eta = \eta(\varepsilon, k)$ such that uniformly in $2 \leqslant R \leqslant P^\eta$ one has the estimate

$$\sup_{\alpha\in\mathfrak{n}} |f(\alpha;P,R)| \ll P^{1-\rho(k)+\varepsilon},$$

where $\rho(k)^{-1} = k(\log k + O(\log\log k))$. Our recent work [3] on Waring's problem yields progress on moment estimates for the Weyl sum $f(\alpha;P,R)$ that makes it possible to refine this bound, and make it more explicit.

Theorem 1.1. *Let $k \geqslant 6$, and define the positive number $\rho(k)$ by*

$$\rho(k)^{-1} = k(\log k + 8.02113). \tag{1.2}$$

Then, there is a positive number $\eta = \eta(k)$ with the property that uniformly in $2 \leqslant R \leqslant P^\eta$ one has the estimate

$$\sup_{\alpha\in\mathfrak{n}} |f(\alpha;P,R)| \ll P^{1-\rho(k)}.$$

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This new bound implies an improvement of [13, Theorem 1.2] concerned with localised estimates for the fractional parts of αn^k.

Theorem 1.2. *Suppose that $\alpha \in \mathbb{R}$. Let $k \geqslant 6$ and define $\rho(k)$ via* (1.2)*. Then, whenever N is sufficiently large in terms of k, one has*

$$\min_{1 \leqslant n \leqslant N} \|\alpha n^k\| \leqslant N^{-\rho(k)}.$$

Here, as is usual in this context, we write $\|\theta\|$ for $\min\{|\theta - n| : n \in \mathbb{Z}\}$. For comparison, a similar conclusion is provided by [13, Theorem 1.2] with $\rho(k)^{-1} = k(\log k + O(\log\log k))$. The numerical values for permissible $\rho(k)$ may be improved for small values of k. We direct the reader to Theorems 4.1 and 5.1 for explicit such refinements valid for $6 \leqslant k \leqslant 20$. These conclusions are superior to all estimates hitherto available when $k \geqslant 10$.

Our proof of Theorem 1.1 draws inspiration from the second author's earlier work [12, 13], but also imports our more recent ideas through an estimate of Weyl's type that occurs as [4, Theorem 3.5]. This bound is most powerful when the argument α is close to a fraction a/q with $(a, q) = 1$ and q is of rough size $P^{k/2}$. This results in genuinely improved performance of the overall infrastructure underlying the proof of [13, Theorem 1.1]. In addition, we improve the large sieve estimate embodied in [13, Section 4]. The large sieve is replaced by a more direct use of the Sobolev-Gallagher inequality to remove an unwanted restriction to even moments of smooth Weyl sums in the final estimate ([13, Lemma 4.1]). Having achieved the latter, we use the occasion to supply admissible exponents for moments of order t, with $t > 4$ a real, not necessarily even number. This result, Theorem 2.1 below, will prove useful in applications of major arcs moment estimates, as once again the restriction to even moments in optimisation procedures like those in [6, Section 8] or [5, Section 6] is certainly undesired, typically accommodated *a posteriori*, and now removable, at least for larger k.

2. Admissible exponents

Our goal in this section is to establish estimates for moments of smooth Weyl sums of sufficient flexibility that technical complications in our later applications may be avoided. For the remainder of the paper, we fix a natural number $k \geqslant 2$. Recall the definition (1.1), and define the moment

$$U_t(P, R) = \int_0^1 |f(\alpha; P, R)|^t \, \mathrm{d}\alpha,$$

where t is a non-negative real number. Following earlier convention, we say that the real number Δ_t is *admissible* (for t) when, for each $\varepsilon > 0$, there exists $\eta > 0$ having the property that, whenever $2 \leqslant R \leqslant P^\eta$, one has

$$U_t(P, R) \ll P^{t-k+\Delta_t+\varepsilon}.$$

We note that admissible exponents Δ_t are non-negative and may be chosen so that $\Delta_t \leqslant k$.

In early work on moments of smooth Weyl sums admissible exponents are denoted differently. Since the discussion was focussed on moments of order $t = 2s$ with $s \in \mathbb{N}$, the subscript of the exponent was often s, not $2s$, which would be in line with our definition. The reader should keep this in mind when comparing our findings with earlier results.

In order to simplify our exposition, we henceforth adopt the following convention concerning ε, R and η. First, whenever ε occurs in a statement, we assert that the statement holds for each positive value of ε. Implicit constants hidden in the symbols of Vinogradov and Landau may depend on the value assigned to ε. Second, should R or η appear in a statement, then it is asserted that the statement holds whenever $R \leqslant P^\eta$ and η is taken to be a positive number sufficiently small in terms of ε.

For each non-negative number t, we define the positive number δ_t to be the unique solution of the equation

$$\delta_t + \log \delta_t = 1 - t/k. \tag{2.1}$$

The conclusion of [12, Theorem 2.1] shows that when $k \geqslant 4$ and t is an even integer with $t \geqslant 4$, then the exponent $\Delta_t = k\delta_t$ is admissible. Note that our earlier cautionary comment applies, as the quantity $\delta_{s,k}$ occurring in the statement of [12, Theorem 2.1] is equal to our δ_{2s}. Moreover, when $t = 2s$ with s a natural number, then it follows via orthogonality that $U_t(P,R)$ is equal to the number of solutions of the equation

$$x_1^k + \cdots + x_s^k = y_1^k + \cdots + y_s^k, \tag{2.2}$$

with $x_i, y_i \in \mathscr{A}(P,R)$, a quantity which in [12] is denoted $S_s(P,R)$. We now extend this earlier result to the situation in which t is permitted to be any real number with $t \geqslant 4$.

Theorem 2.1. *Let $k \geqslant 6$ and suppose that t is a real number with $t \geqslant 4$. Then the exponent $\Delta_t = k\delta_t$ is admissible.*

This is a special case of a more general theorem in which a weighted analogue of the exponential sum $f(\alpha; P, R)$ appears. When $w(n) \in \mathbb{C}$ $(n \in \mathbb{N})$, we define

$$\|w\|_X = \max_{1 \leqslant n \leqslant X} |w(n)|.$$

Also, when t is a positive number, put

$$U_t(P,R;w) = \int_0^1 \Bigl| \sum_{n \in \mathscr{A}(P,R)} w(n) e(\alpha n^k) \Bigr|^t \, \mathrm{d}\alpha. \tag{2.3}$$

We say that the real number Δ_t is *weight-uniform admissible* when, for each $\varepsilon > 0$, there exists $\eta > 0$ having the property that, whenever $2 \leqslant R \leqslant P^\eta$, uniformly in w one has

$$U_t(P,R;w) \ll \|w\|_P^t \, P^{t-k+\Delta_t+\varepsilon}. \tag{2.4}$$

Note that when s is a natural number, it follows by orthogonality that

$$U_{2s}(P,R;w) = \sum_{\mathbf{x},\mathbf{y}} \prod_{i=1}^s w(x_i)\overline{w(y_i)},$$

where $x_i, y_i \in \mathscr{A}(P,R)$ $(1 \leqslant i \leqslant s)$ are constrained by (2.2). Thus, one has

$$U_{2s}(P,R;w) \leqslant \|w\|_P^{2s} \, U_{2s}(P,R), \tag{2.5}$$

and hence the exponent Δ_{2s} is weight-uniform admissible whenever it is admissible.

Theorem 2.2. *Let $k \geqslant 6$ and suppose that t is a real number with $t \geqslant 4$. Then the exponent $\Delta_t = k\delta_t$ is weight-uniform admissible.*

Proof. Consider a natural number $s \geqslant 2$, and define the positive number Δ_{2s} to be the unique solution of the equation

$$\frac{\Delta_{2s}}{k} + \log \frac{\Delta_{2s}}{k} = 1 - \frac{2s}{k} - \frac{5}{16k^2}. \tag{2.6}$$

We assert that the exponent Δ_{2s} is admissible. In order to confirm this assertion, observe first that when $s = 2$ the exponent $k - 2$ is admissible, as a consequence of Hua's lemma (see [9, Lemma 2.5]). Moreover, one has

$$1 - \frac{2}{k} + \log\Big(1 - \frac{2}{k}\Big) < 1 - \frac{4}{k} - \frac{2}{k^2} < 1 - \frac{4}{k} - \frac{5}{16k^2}.$$

Hence, the exponent Δ_4 defined via (2.6) is admissible. When s is a natural number exceeding 2, meanwhile, it follows from the proof of [12, Theorem 2.1] that an admissible exponent Δ for $2s$ exists satisfying

$$\frac{\Delta}{k} + \log \frac{\Delta}{k} \leqslant \delta_{2s-2} + \log \delta_{2s-2} - \frac{2}{k} + \frac{E}{2k^2},$$

where $E \leqslant k\, 2^{2-k} - 1 \leqslant -5/8$. This upper bound is a direct interpretation of the penultimate displayed equation of the proof of [12, Theorem 2.1], on page 167, with the inequality for E immediately following the latter equation. In view of (2.1), one has

$$\frac{\Delta}{k} + \log \frac{\Delta}{k} \leqslant 1 - \frac{2s}{k} - \frac{5}{16k^2},$$

and so a comparison with (2.6) confirms that Δ_{2s} is admissible.

Given a real number t with $t \geqslant 4$, we put $s = \lfloor t/2 \rfloor$ and $v = t/2 - s$, so that $t = 2s + 2v$. An application of Hölder's inequality leads from (2.3) via (2.5) to the bound

$$U_t(P, R; w) \leqslant \|w\|_P^t\, U_{2s}(P, R)^{1-v}\, U_{2s+2}(P, R)^v. \tag{2.7}$$

We have now effectively removed the weights from consideration.

We make use of the admissible exponent Δ_{2s} defined via (2.6), but refine slightly the admissible exponent Δ_{2s+2}. Let $\omega = 2^{1-k}(1 - \Delta_{2s}/k)$. The argument of the proof of [12, Theorem 2.1] on page 167 shows that the positive number

$$\Delta'_{2s+2} = \Delta_{2s}\Big(1 - \frac{2 - \omega}{k + \Delta_{2s}}\Big)$$

is admissible for $2s + 2$. In view of the upper bound (2.4), we find from (2.7) that Δ_t is weight-uniform admissible, where

$$\begin{aligned} \Delta_t &= (1 - v)\Delta_{2s} + v\Delta'_{2s+2} \\ &= (1 - v)\Delta_{2s} + v\Delta_{2s}\Big(1 - \frac{2 - \omega}{k + \Delta_{2s}}\Big) \\ &= \Delta_{2s}\Big(1 - v\,\frac{2 - \omega}{k + \Delta_{2s}}\Big). \end{aligned}$$

Observe that

$$\frac{\Delta_t}{k} + \log \frac{\Delta_t}{k} = \frac{\Delta_{2s}}{k} + \log \frac{\Delta_{2s}}{k} - v\,\frac{\Delta_{2s}}{k}\,\frac{2 - \omega}{k + \Delta_{2s}} + \log\Big(1 - v\frac{2 - \omega}{k + \Delta_{2s}}\Big),$$

so that from (2.6) we obtain

$$\frac{\Delta_t}{k}+\log\frac{\Delta_t}{k}\leqslant 1-\frac{2s}{k}-\frac{5}{16k^2}-v\,\frac{\Delta_{2s}}{k}\,\frac{2-\omega}{k+\Delta_{2s}}-v\,\frac{2-\omega}{k+\Delta_{2s}}-\frac{v^2(2-\omega)^2}{2(k+\Delta_{2s})^2}.$$

Since

$$2-\omega=2-2^{1-k}(1-\Delta_{2s}/k)\geqslant 2-(1-\Delta_{2s}/k)=1+\Delta_{2s}/k,$$

we deduce that

$$\frac{\Delta_t}{k}+\log\frac{\Delta_t}{k}\leqslant 1-\frac{t}{k}+\frac{E}{2k^2},$$

where

$$E=-\frac{5}{8}+2kv\omega-v^2.$$

Recalling that $k\geqslant 6$ and $0\leqslant v\leqslant 1$, we have

$$E\leqslant -\frac{5}{8}+v(k\,2^{2-k}-v)<0,$$

and so

$$\frac{\Delta_t}{k}+\log\frac{\Delta_t}{k}\leqslant 1-\frac{t}{k}.$$

In view of relation (2.1), it follows that the exponent $k\delta_t$ is weight-uniform admissible for t. This proves Theorem 2.2, and the conclusion of Theorem 2.1 follows as a corollary. □

3. A new upper bound for smooth Weyl sums

We adapt the arguments of [13] so as to obtain a new minor arc estimate for the smooth Weyl sum $f(\alpha;P,R)$. We begin with an analogue of [13, Lemma 4.1]. The strategy adopted in the proof of the latter makes use of the large sieve inequality to estimate an exponential sum stemming from an even power of $f(\alpha;P,R)$. Here, we adopt a more flexible approach, able to handle positive real powers of this exponential sum, by appealing directly to the Sobolev-Gallagher inequality.

Lemma 3.1. *Suppose that $1/2<\lambda<1$, and write $M=P^\lambda$. Let $\alpha\in\mathbb{R}$, and suppose that $a\in\mathbb{Z}$ and $q\in\mathbb{N}$ satisfy*

$$(a,q)=1,\quad |q\alpha-a|\leqslant \tfrac{1}{2}(MR)^{-k},\quad q\leqslant 2(MR)^k,$$

and either $|q\alpha-a|>MP^{-k}$ or $q>MR$. Then, if t is a real number with $t>k+1$ and Δ_t is weight-uniform admissible, one has

$$f(\alpha;P,R)\ll M^{1+\varepsilon}+P^{1+\varepsilon}\big(M^{-1}(P/M)^{\Delta_t}(1+q(P/M)^{-k})\big)^{1/t}.$$

Proof. We begin our argument in the same manner as the proof of [13, Lemma 4.1]. Define the set

$$\mathscr{B}(M,\varpi,R)=\{v\in\mathscr{A}(P,R)\cap(M,MR]:\varpi|v,\text{ and }p|v\text{ implies }p\geqslant\varpi\},$$

in which, here and henceforth, the letters ϖ and p both denote prime numbers. Then, from [13, equation (4.1)], we find that there exists an integer d with $1\leqslant d\leqslant P/M$, a real number $\theta\in[0,1)$, and a prime number ϖ with $\varpi\leqslant R$, such that

$$f(\alpha;P,R)\ll M^{1+\varepsilon}+P^\varepsilon Rg(\alpha;d,\varpi,\theta),\tag{3.1}$$

where

$$g(\alpha;d,\varpi,\theta)=\sum_{\substack{v\in\mathscr{B}(M/d,\varpi,R)\\(v,q)=1}}\biggl|\sum_{\substack{u\in\mathscr{A}(P/M,\varpi)\\(u,q)=1}}e(\alpha(uvd)^k+\theta u)\biggr|.$$

The argument of the proof of [13, Lemma 4.1] shows that one can partition the integers $v\in\mathscr{B}(M/d,\varpi,R)$ with $(v,q)=1$ into L classes $\mathscr{V}_1,\ldots,\mathscr{V}_L$, with $L=O(q^\varepsilon d^k)$, having the following property. That is, for each j, as v varies over $\mathscr{V}_j$, the real numbers $\alpha(vd)^k$ are spaced apart at least $\xi=\frac{1}{2}\min\{q^{-1},(P/M)^{-k}\}$ modulo 1. Define

$$h(\alpha)=\sum_{\substack{u\in\mathscr{A}(P/M,\varpi)\\(u,q)=1}}e(\alpha u^k+\theta u).$$

Then an application of the Sobolev-Gallagher inequality (see [7, Lemma 1.1]) to the continuously differentiable function $|h(\beta)|^t$ reveals that for $1\leqslant j\leqslant L$, one has

$$\sum_{v\in\mathscr{V}_j}|h(\alpha(vd)^k)|^t\ll\xi^{-1}\int_0^1|h(\beta)|^t\,\mathrm{d}\beta+\int_0^1|h(\beta)^{t-1}h'(\beta)|\,\mathrm{d}\beta. \tag{3.2}$$

Define the weights

$$w_1(n)=\begin{cases}e(\theta n), & \text{when } (n,q)=1,\\ 0, & \text{when } (n,q)>1,\end{cases}$$

and $w_2(n)=2\pi i n^k w_1(n)$. Then it follows from (2.4) that, for each prime $\varpi\leqslant R$, one has

$$\int_0^1|h(\beta)|^t\,\mathrm{d}\beta=U_t(P/M,\varpi;w_1)\ll(P/M)^{t-k+\Delta_t+\varepsilon}$$

and

$$\int_0^1|h'(\beta)|^t\,\mathrm{d}\beta=U_t(P/M,\varpi;w_2)\ll(P/M)^{kt}(P/M)^{t-k+\Delta_t+\varepsilon}.$$

An application of Hölder's inequality therefore leads from (3.2) to the bound

$$\begin{aligned}\sum_{v\in\mathscr{V}_j}|h(\alpha(vd)^k)|^t&\ll\xi^{-1}(P/M)^{t-k+\Delta_t+\varepsilon}+(P/M)^k(P/M)^{t-k+\Delta_t+\varepsilon}\\&\ll\bigl(q+(P/M)^k\bigr)(P/M)^{t-k+\Delta_t+\varepsilon}.\end{aligned}$$

By summing these contributions from each set $\mathscr{V}_j$ for $1\leqslant j\leqslant L$, we thus obtain

$$\sum_{\substack{v\in\mathscr{B}(M/d,\varpi,R)\\(v,q)=1}}|h(\alpha(vd)^k)|^t\ll q^\varepsilon d^k(P/M)^{t+\Delta_t+\varepsilon}(1+q(P/M)^{-k}).$$

An application of Hölder's inequality to (3.1) therefore confirms that

$$|f(\alpha;P,R)|^t\ll M^{t+\varepsilon}+P^\varepsilon d^k\bigl(1+q(P/M)^{-k}\bigr)(M/d)^{t-1}(P/M)^{t+\Delta_t},$$

whence, since $t>k+1$, we obtain the bound asserted in the statement of the lemma. □

Within the arguments to follow, we work with major arcs of various formats. Thus, when $1 \leqslant Q \leqslant P^{k/2}$, let $\mathfrak{M}(Q)$ denote the union of the intervals

$$\{\alpha \in [0,1) : |q\alpha - a| \leqslant QP^{-k}\}$$

with $0 \leqslant a \leqslant q \leqslant Q$ and $(q,a) = 1$. We write $\mathfrak{m}(Q) = [0,1) \setminus \mathfrak{M}(Q)$ and $\mathfrak{m} = \mathfrak{m}(P)$. Note that $\mathfrak{m} = \mathfrak{n} \cap [0,1)$.

We next extract a minor arc estimate for smooth Weyl sums from Lemma 3.1. Given a family $(\Delta_s)_{s>0}$ of weight-uniform admissible exponents, we define the real number $\tau = \tau(k)$ by means of the relation

$$\tau(k) = \max_{w \in \mathbb{N}} \frac{k - 2\Delta_{2w}}{4w^2}.$$

We have observed already that for $w \in \mathbb{N}$, any admissible exponent Δ_{2w} is also weight-uniform admissible. Thus, as discussed in the preamble to [3, equation (5.1)] or [4, Lemma 3.1], one finds that $\tau(k) \leqslant 1/(4k)$. This exponent is relevant to uniform estimates of Weyl type for $f(\alpha; P, R)$.

Lemma 3.2. *Suppose that $k \geqslant 2$. Then, uniformly in $1 \leqslant Q \leqslant P^{k/2}$, one has the bound*

$$\sup_{\alpha \in \mathfrak{m}(Q)} |f(\alpha; P, R)| \ll PQ^{\varepsilon - 2\tau(k)/k}.$$

In particular, writing $D = 4.5139506$, one has

$$\sup_{\alpha \in \mathfrak{m}(Q)} |f(\alpha; P, R)| \ll PQ^{-1/(Dk^2)}.$$

Proof. The respective conclusions are available from [4, Lemma 3.3] and [4, Lemma 3.4]. □

In order to state the next theorem, we introduce the real number $\sigma = \sigma(k)$, defined via the relation

$$\sigma(k)^{-1} = \inf_{t > k+1} \left(t + \frac{1 + \Delta_t}{2\tau(k)} \right). \tag{3.3}$$

We then define the associated quantity $\lambda = \lambda(k)$ by putting

$$\lambda(k) = 1 - \frac{\sigma(k)}{2\tau(k)}. \tag{3.4}$$

Theorem 3.3. *Suppose that $1/2 < \lambda < 1$. Then one has*

$$\sup_{\alpha \in \mathfrak{m}(P^\lambda R)} |f(\alpha; P, R)| \ll P^{1 - \sigma(k) + \varepsilon}.$$

Proof. We put $M = P^\lambda$ and apply Lemma 3.1. By Dirichlet's approximation theorem, there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with

$$(a,q) = 1, \quad q \leqslant 2(MR)^k \quad \text{and} \quad |q\alpha - a| \leqslant \tfrac{1}{2}(MR)^{-k}.$$

When $\alpha \in \mathfrak{m}(P^\lambda R)$, it follows that either $q > MR$ or $|q\alpha - a| > MRP^{-k}$, and hence Lemma 3.1 shows that for each $t > k + 1$ and each weight-uniform admissible exponent

Δ_t, one has

$$\begin{aligned} f(\alpha;P,R) &\ll P^{\lambda+\varepsilon}+P^{1+\varepsilon}\big(P^{-\lambda+(1-\lambda)\Delta_t}(1+qP^{-k(1-\lambda)})\big)^{1/t}\\ &\ll P^{1-\sigma}+P^{1+\varepsilon}\big(P^{-1+\sigma(1+\Delta_t)/(2\tau)}(1+qP^{-k\sigma/(2\tau)})\big)^{1/t}. \end{aligned} \tag{3.5}$$

Observe that from (3.3) one has

$$\sup_{t>k+1}\frac{1}{t}\left(\frac{1}{\sigma}-\frac{1+\Delta_t}{2\tau}\right)\geqslant 1,$$

whence

$$\sup_{t>k+1}\frac{1}{t}\left(1-\frac{\sigma(1+\Delta_t)}{2\tau}\right)\geqslant \sigma.$$

Hence, we deduce from (3.5) that when $q\leqslant P^{k\sigma/(2\tau)}$, one has

$$f(\alpha;P,R)\ll P^{1-\sigma+\varepsilon}. \tag{3.6}$$

It remains to handle the situation in which $q>P^{k\sigma/(2\tau)}$. Write $Q=\frac{1}{2}R^{-k}P^{k\sigma/(2\tau)}$, and note that one then has

$$\tfrac{1}{2}(MR)^{-k}=\tfrac{1}{2}R^{-k}\big(P^{1-\sigma/(2\tau)}\big)^{-k}=QP^{-k}.$$

Thus, we see that $\alpha\in\mathfrak{m}(Q)$, and hence Lemma 3.2 delivers the bound

$$f(\alpha;P,R)\ll PQ^{\varepsilon-2\tau/k}\ll P^{1-\sigma+\varepsilon}R. \tag{3.7}$$

In view of our conventions concerning ε and R, the conclusion of the theorem follows on combining (3.6) and (3.7). □

4. The proof of Theorem 1.1

The first goal of this section is to optimise parameters in Theorem 3.3 so as to prove Theorem 1.1.

The proof of Theorem 1.1. We assume throughout that $k\geqslant 6$. The second conclusion of Lemma 3.2 shows that one can proceed using the value $\tau=1/(2Dk)$. Also, as a consequence of Theorem 2.2, the exponent $\Delta_t=k\delta_t$ is weight-uniform admissible for $t\geqslant 4$, with δ_t defined by equation (2.1). In particular, one has $\Delta_t\leqslant k\mathrm{e}^{1-t/k}$. Thus, the exponent σ defined in (3.3) satisfies the bound

$$\sigma^{-1}\leqslant\inf_{t>k+1}\big(t+Dk(1+k\mathrm{e}^{1-t/k})\big).$$

One may verify that the infimum on the right hand side here is attained when $t=k\log k+k(1+\log D)$, and thus

$$\begin{aligned} \sigma^{-1}&\leqslant k\log k+k(1+\log D)+Dk(1+1/D)\\ &=k\log k+k(D+2+\log D). \end{aligned}$$

We therefore conclude that one has $\sigma^{-1}\leqslant k(\log k+\phi)$, where $\phi=8.0211233\ldots$. We have now proved that

$$\sup_{\alpha\in\mathfrak{m}(P^\lambda R)}|f(\alpha;P,R)|\ll P^{1-\sigma+\varepsilon}.$$

Here λ is the number defined in (3.4). Since $P^\lambda R\leqslant P$ and f has period 1, this establishes a little more than is actually claimed in Theorem 1.1. □

By making use of the tables of admissible exponents to be found in [10] ($k=6$) and [11] ($7\leqslant k\leqslant 20$), one may numerically compute the value of the exponent $\sigma(k)$ defined via (3.3). In the table below, we record values of $2w$, and a corresponding admissible exponent Δ_{2w}. These numbers are taken from [10, 11] where the numbers $\lambda_w=2w-k+\Delta_{2w}$ are tabulated. We also supply an upper bound for the number $T(k)$ having the property that $\tau(k)>T(k)^{-1}$. These data have been computed for $6\leqslant k\leqslant 13$ directly from the definition, and are tabulated for $14\leqslant k\leqslant 20$ in [3]. Here, we note a typographic error in the latter source. Thus, the heading w in the second column of [3, Table 2] should read $2w$ (in place of w). Finally, we report numbers $S(k)$ having the property that $\sigma(k)>S(k)^{-1}$. All figures are rounded up in the last digit displayed. We summarise these conclusions in the form of a theorem.

Theorem 4.1. *When* $6\leqslant k\leqslant 20$*, one has the bound*

$$\sup_{\alpha\in\mathfrak{m}}|f(\alpha;P,R)|\ll P^{1-\sigma(k)+\varepsilon},$$

where $\sigma(k)>S(k)^{-1}$.

k	$2w$	Δ_{2w}	$T(k)$	t	Δ_t	$S(k)$
6	10	1.724697	39.2064	22	0.086042	43.2899
7	12	2.014382	48.4647	26	0.192538	54.8980
8	14	2.310600	58.0088	32	0.189117	66.4897
9	16	2.603928	67.5080	38	0.190186	78.1736
10	18	2.894572	76.9440	44	0.192696	89.8855
11	20	3.184973	86.3921	48	0.241313	101.6199
12	22	3.470081	95.6553	54	0.239541	113.2844
13	24	3.755717	104.9455	60	0.239277	125.0283
14	26	4.039939	114.1869	66	0.240167	136.8055
15	28	4.323087	123.3903	74	0.209471	148.6185
16	30	4.606286	132.5981	80	0.213791	160.4732
17	32	4.888677	141.7763	86	0.218395	172.3698
18	34	5.170691	150.9411	92	0.223249	184.3193
19	36	5.451758	160.0695	98	0.228287	196.3057
20	38	5.732224	169.1748	104	0.233496	208.3383

TABLE 1. Choice of parameters for $6\leqslant k\leqslant 20$.

When $k\geqslant 10$, the bounds supplied by Theorems 1.1 and 4.1 are superior to any previously available bound of Weyl's type for either smooth or classical Weyl sums. When $k\leqslant 9$, however, the bound

$$\sup_{\alpha\in\mathfrak{m}}|f(\alpha;P,P)|\ll P^{1-\frac{1}{k(k-1)}+\varepsilon},$$

available via recent developments in Vinogradov's mean value theorem (see [2, 14]), provides superior exponents for classical Weyl sums. For smooth exponential sums, meanwhile, the estimates for $\sigma(k)$ in the table are still superior to those listed in [11].

5. The fractional part of αn^k

The proof of Theorem 1.2 is achieved via a pedestrian modification of [13, §6], by utilising Theorem 3.3. Let $\nu = (\sigma(k) - \rho(k))/2$, so that $0 < \nu < \sigma(k)$. Then the only issue to check is that, with $H = P^{\sigma(k)-\nu}$, one has

$$(HP^{\lambda(k)}R)^{k-1}P^{\lambda(k)-k} \ll P^{-\sigma(k)}.$$

However, in view of (3.4) and the bound $\tau(k) \leqslant 1/(4k)$ noted in the preamble to Lemma 3.2, one has

$$(k-1)\sigma(k) + k\lambda(k) - k = \left(k - 1 - \frac{k}{2\tau(k)}\right)\sigma(k) < -2\sigma(k).$$

With plenty of room to spare, this suffices to confirm the validity of the argument corresponding to [13, §6], and we find that

$$\min_{1\leqslant n\leqslant N} \|\alpha n^k\| \ll N^{\nu-\sigma(k)}.$$

Since $\sigma(k) > \rho(k)$, the desired conclusion follows from the definition of ν.

By applying the same argument as described above for $10 \leqslant k \leqslant 20$, using the explicit exponents calculated in the previous section as recorded in the table therein, one obtains the following conclusion.

Theorem 5.1. *Let k be an integer with $10 \leqslant k \leqslant 20$. Then, with the exponent $S(k)$ defined as in Table 1, one has*

$$\min_{1\leqslant n\leqslant N} \|\alpha n^k\| \ll N^{-1/S(k)}.$$

This theorem improves on the earlier results of [11] for $k \geqslant 10$. Such conclusions are also addressed by Baker in the discussion following the statement of [1, Theorem 3]. Our new conclusions recorded in Theorems 1.2 and 5.1 improve on the estimates recorded in part (ii) of the latter discussion for $k \geqslant 10$. In [1, Theorem 2], Baker points out (*inter alia*) that the new conclusions available from recent breakthroughs on Vinogradov's mean value theorem (see [2, 14], for example) yield the upper bound

$$\min_{1\leqslant n\leqslant N} \|\alpha n^k\| \ll N^{\varepsilon-1/(k(k-1))}.$$

On noting that the exponent $S(k)$ recorded in Table 1 exceeds $k(k-1)$ for $k \leqslant 9$, we see that our new estimates are superior to those available via this progress on Vinogradov's mean value theorem only for $k \geqslant 10$.

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