The Tri-Pants Graph

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UVA Topology REU

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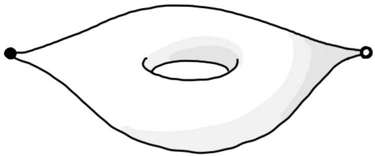
Introduction

- This work was developed during the Topology REU at the University of Virginia in the summer of 2021.
- We consider the Tri-pants graph as introduced by Maloni and Palesi.
- The tri-pants graph $\mathcal{TP}$ is a combinatorial graph defined in terms of tri-pants on the twice punctured torus.
- $\mathcal{TP}$ is higher complexity analogue to the dual of the Farey complex, $\mathcal{F}^*$.
- Making use of the connections between $\mathcal{TP}$ and $\mathcal{F}^*$, we prove that the Tri-pants graph is connected and has infinite diameter.

Definition

*The **twice-punctured torus** is obtained by removing two points (also called punctures) from a compact surface with genus 1. We denote it as* $T^2 \setminus \{\circ, \bullet\}$.



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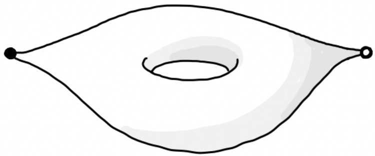

Definition

A closed curve α *is* ***separating*** *in a surface, S if* $S \setminus \alpha$ *is disconnected.*

A simple closed curve is said to be ***essential*** *if it is non-trivial, not homotopic to a puncture or boundary.*

Pants Decompositions

A **pair of pants** is a surface with genus 0 and 3 boundary components/punctures.

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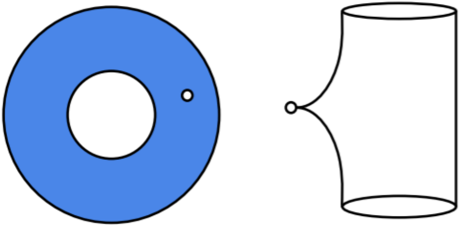

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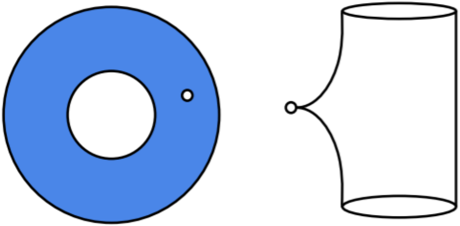

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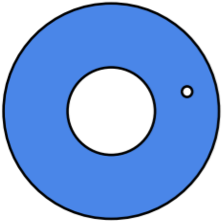

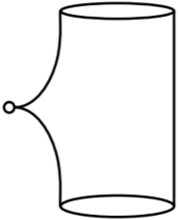

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Under these assumptions, $(T^2 \setminus \{\circ, \bullet\}) \setminus (\alpha \cup \alpha')$ is the union of two punctured annuli.

Tri-Pant

Definition (Tri-Pant)

A ***tri-pant*** $T = \{\alpha, \alpha', \beta, \beta', \gamma, \gamma'\}$ *of* $T^2 \setminus \{\circ, \bullet\}$ *is a collection of 6 simple closed curves (up to homotopy) so that*

- *Each curve is essential and non-separating.*
- $\{\alpha, \alpha'\}$, $\{\beta, \beta'\}$, *and* $\{\gamma, \gamma'\}$ *all describe pants decompositions.*
- *Every pair of curves in* T *intersect exactly once, unless they determine a pants decomposition.*

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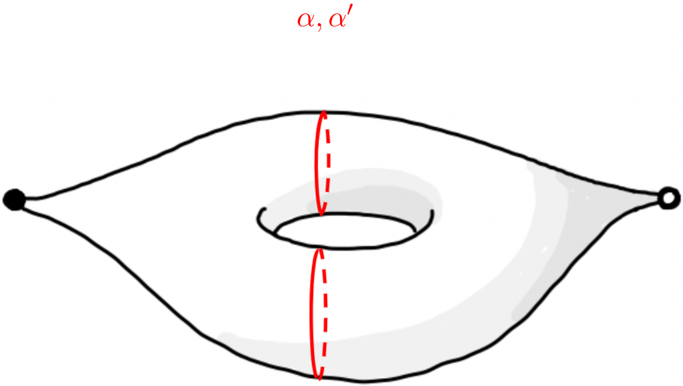
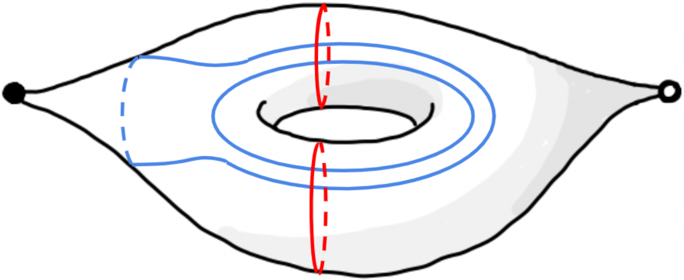
An Alternate Definition: A *tri-pant* of $T^2 \setminus \{\circ, \bullet\}$ is a maximal collection of essential non-separating simple closed curves in $T^2 \setminus \{\circ, \bullet\}$ that intersect pairwise at most once.

Example of a Tri-Pant



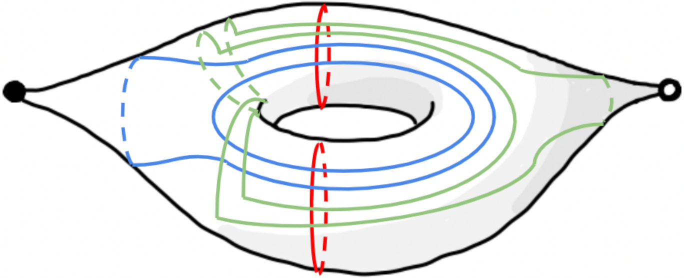
Example of a Tri-Pant

$\color{red}{\alpha, \alpha'}$, $\color{blue}{\beta, \beta'}$



Example of a Tri-Pant

α, α', β, β', γ, γ'



Tri-Arcs

Definition

*A **tri-arc** is a collection of three distinct simple, essential, and unoriented arcs* a, b, c *belonging to the set*

$$\mathcal{A}_\bullet := \{\textit{isotopy classes of arcs with both endpoints at } \bullet\}$$

that pairwise intersect only at $\bullet$ *and do not bound a cylinder.*

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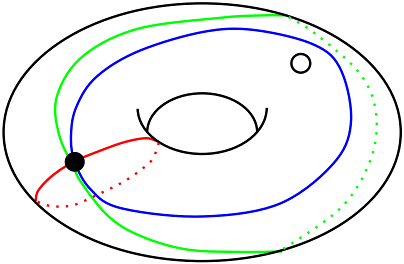
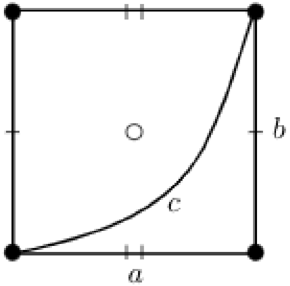


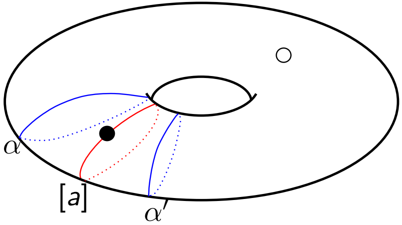


Tri-Arcs (cont.)

Lemma

There is a bijection between tri-pants and tri-arcs.

How to visualize: let $[a] \in \mathcal{A}$. Construct a tubular neighborhood N_a about $[a]$ and denote the boundaries of N_a to be α and α'. Then $\{\alpha, \alpha'\}$ forms a pants decomposition on $T^2 \setminus \{\circ, \bullet\}$.



**We can obtain $[a]$ from $\{\alpha, \alpha'\}$ in a similar way.

Elementary Moves

We say two tri-pants T, T' *differ by an elementary move* if the corresponding tri-arcs T_*, T'_* differ in one of the two following ways:

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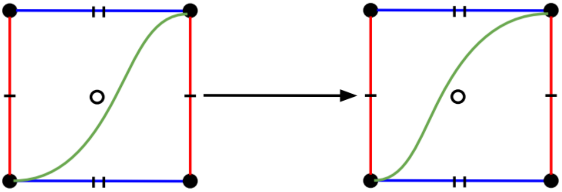

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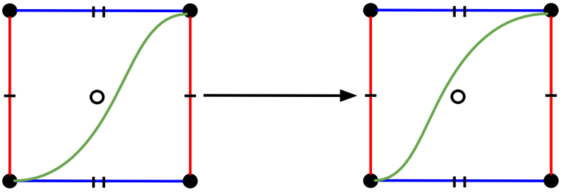

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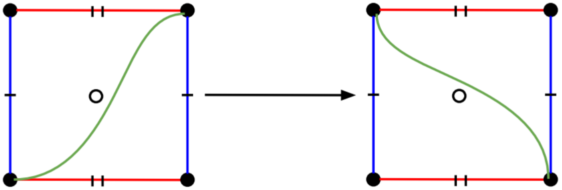

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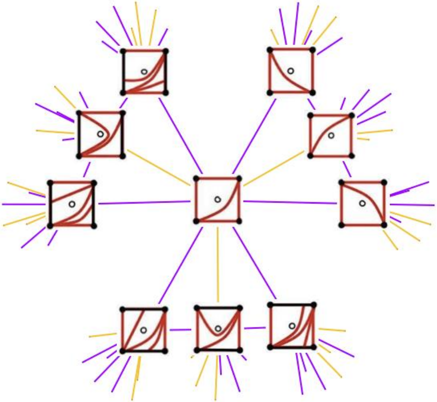
Definition (Tri-Pants Graph)

*The **tri-pants graph** is a graph $\mathcal{TP}$ with vertices corresponding to distinct tri-pants on $T^2 \setminus \{\circ, \bullet\}$. Two vertices are connected by an edge if the associated tri-pants differ by an elementary move.*



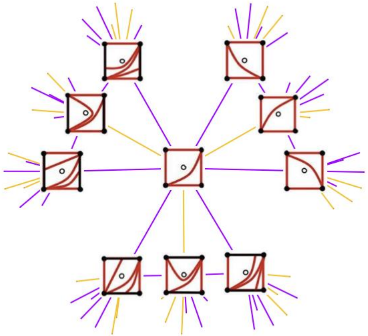
We will show:

- Connectedness
- Infinite diameter

Proposition

Let $T_* = \{[a], [b], [c]\} \in \mathcal{TP}$. *Then* $\deg(T_*) = 9$.

By cutting and gluing, there are three ways to represent T_* (with a, b, and c as the diagonal, respectively). For each representation, there are two possible small flips and one big flip. One can check that the resulting tri-arcs are distinct. □



The Inclusion Map and the Farey Graph

Lemma

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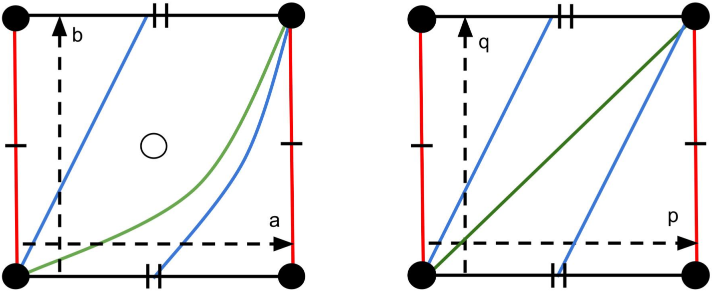
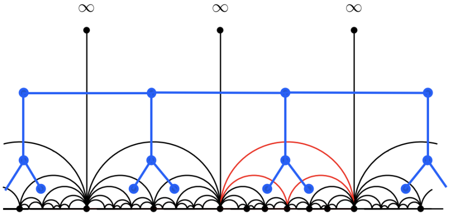
Corollary

The inclusion map $i_* : \pi_1(T^2 \setminus \{\circ\}, \bullet) \longrightarrow \pi_1(T^2, \bullet)$ *sends tri-arcs* T *to triangles* $i(T)$ *in the Farey graph.*



Definition

The dual graph $\mathcal{F}^*$ associated with the Farey Graph $\mathcal{F}$ is the graph whose vertices correspond to the triangles $T \in \mathcal{F}$, and whose edges connect two vertices $v = T^*$, $v' = (T')^* \in \mathcal{F}^*$ if and only if T and T' are adjacent in $\mathcal{F}$.



The Dual of the Farey Graph ($\mathcal{F}^*$)

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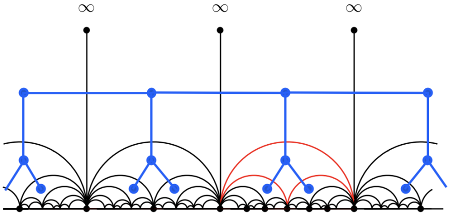

- $\mathcal{F}^*$ is connected and has infinite diameter.
- We examine a connection between $\mathcal{TP}$ and $\mathcal{F}^*$.

The Tri-Pants Graph – The Map π

Definition

Define $\pi : \mathcal{TP} \to \mathcal{F}^*$ *to be a map such that:*

- *Every* $T \in \mathcal{TP}$ *maps to a vertex* $v \in \mathcal{F}^*$ *corresponding to the triangle* $i(T) \in \mathcal{F}$

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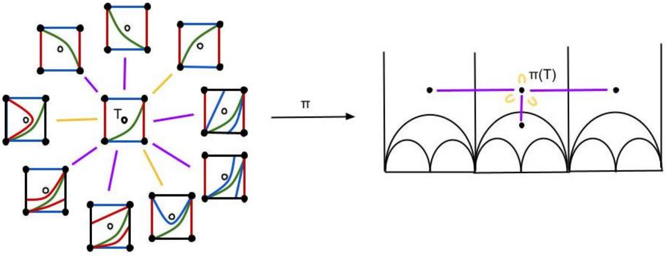

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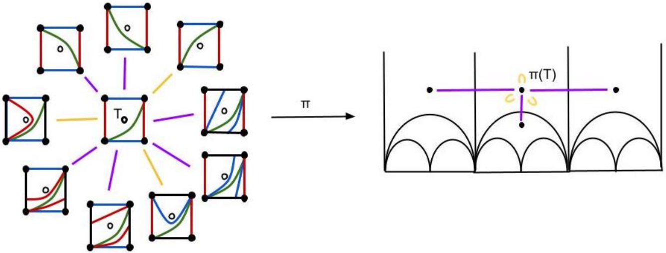

- Define a **fiber** of π to be $\pi^{-1}(v) \in \mathcal{TP}$ for $v \in \mathcal{F}^*$.

The Tri-Pants Graph - Connectedness of the Fibers

Proposition

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Before proving the theorem, we need to present some definitions.

Mapping Class Group

We define the set

$$\mathrm{Homeo}^+(T^2 \backslash \{\circ, \bullet\}) = \{f : T^2 \to T^2 \text{ orientation-preserving homeo.}$$
$$\text{so that } f|_{\{\circ,\bullet\}} = id\}$$

and the **pure mapping class group**

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For homotopy classes $\theta \in \pi_1(T^2 \setminus \{\bullet\}, \circ)$, we can define an element $\text{Push}(\theta) \in \text{PMod}(T^2 \setminus \{\circ, \bullet\})$ which "pushes" $\circ$ along θ, taking all intersecting curves along this path as well

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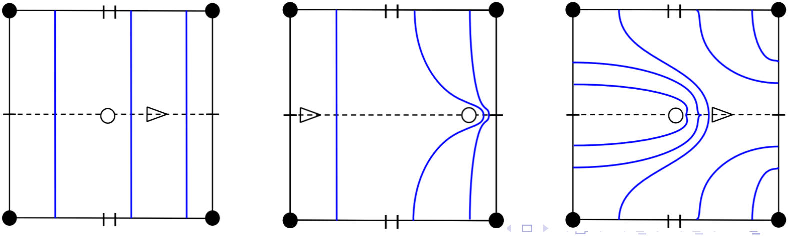
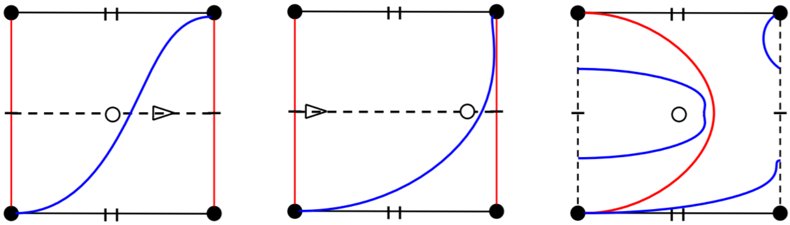
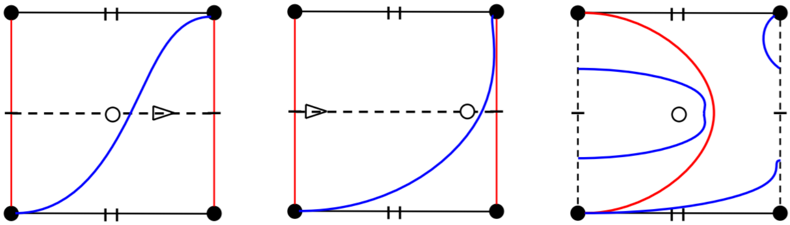




Remark

If θ is parallel to a curve in a tri-arc, then applying the push map corresponds to two big flips



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We may define also the Forget map

$$\mathrm{Forget} : \mathrm{PMod}(T^2 \setminus \{\circ, \bullet\}) \to \mathrm{PMod}(T^2 \setminus \{\bullet\}),$$

which essentially "forgets" the puncture.

Birman's Exact Sequence

$$1 \to \pi_1(T^2 \backslash \{\bullet\}, \circ) \xrightarrow{Push} \mathrm{PMod}(T^2 \backslash \{\circ, \bullet\}) \xrightarrow{Forget} \mathrm{PMod}(T^2 \backslash \{\bullet\}) \to 1$$

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This exact sequence tells us that if $\phi \in \mathrm{PMod}(T^2 \setminus \{\circ, \bullet\})$ is such that Forget$[\phi] = id$, then ϕ is a push map, so $T, \phi(T)$ differ by an even number of big flips.

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Proof (Outline of Proof.)

- Two tri-pants $T, T' \in \pi^{-1}(v) \iff$ their associated tri-arcs $T_* = \{[a], [b], [c]\}, T'_* = \{[a'], [b'], [c']\}$ contain the same homotopy classes of unoriented arcs in T^2, based at $\bullet$.

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- We can find a homeomorphism ϕ of T^2 which fixes $\{\circ, \bullet\}$ and so that $\phi(a) = a'$, $\phi(b) = b'$
  - It turns out that the induced group morphism $\phi_* : \pi_1(T^2, \bullet) \to \pi_1(T^2, \bullet)$ is the identity map.
  - In fact, $\phi_* = id \implies \phi$ is isotopic to the identity in $T^2 \setminus \{\bullet\}$.

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- In either case, T_* and T'_* are separated by a finite sequence of big flips, so there is a path in $\pi^{-1}(v)$ connecting T to T'.

$\square$

The Tri-pants Graph – Connectivity of $\mathcal{TP}$

Theorem

The tri-pants graph $\mathcal{TP}$ is connected.

Let $T, T' \in \mathcal{TP}$. We seek a path in $\mathcal{TP}$ connecting T to T'.

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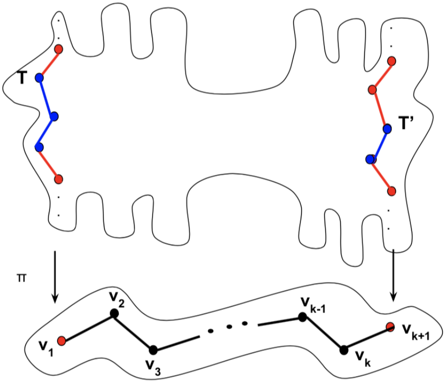
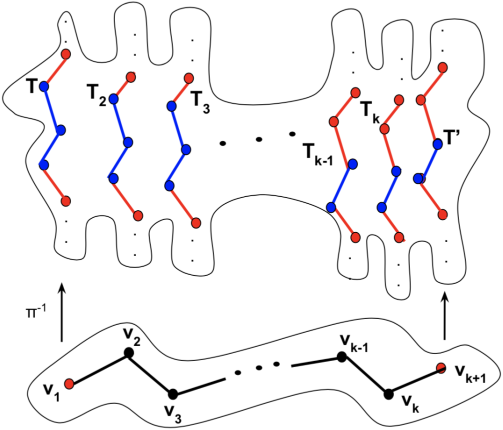


Recall that $\mathcal{F}^*$ is connected. Let $\pi(T) = v_1$ and $\pi(T') = v_{k+1}$. The edges $e_1, \dots, e_k$ create a path along $v_1, \dots, v_{k+1}$ in $\mathcal{F}^*$.

Theorem

The tri-pants graph $\mathcal{TP}$ is connected.

We know that $\pi^{-1}(v_i) \neq \pi^{-1}(v_{i+1})$ for all i.



The Tri-pants Graph – Connectivity of $\mathcal{TP}$

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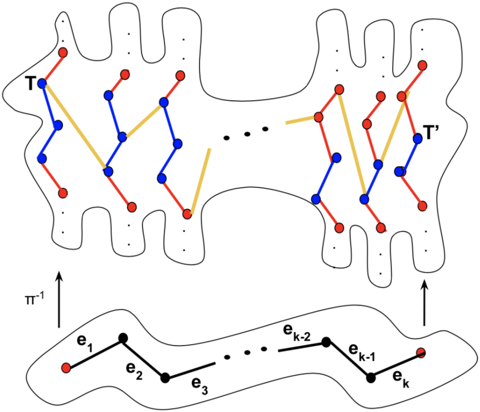
Choose small flips that project to $e_1, \dots, e_k$ under the map π to connect adjacent fibers, similar to takng $\pi^{-1}(e_1, \dots, e_k)$.



The Tri-pants Graph – Connectivity of $\mathcal{TP}$

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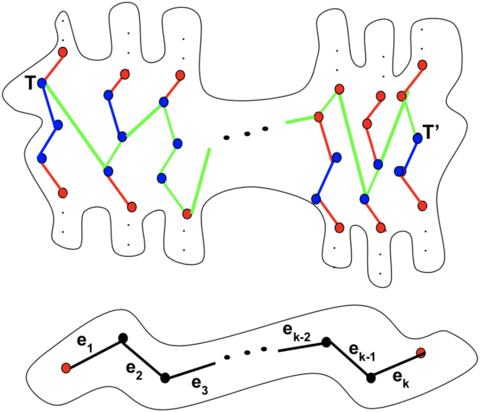
Since each fiber in $\mathcal{TP}$ is connected, we can find a path connecting T and T'. Thus $\mathcal{TP}$ is connected.



The Tri-pants Graph – Infinite Diameter

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Proof.

- By construction of π, for any path $\gamma \in \mathcal{TP}$ connecting T to T', $L(\pi(\gamma)) \leq L(\gamma)$.
- We know that $\mathcal{F}^*$ has infinite diameter, so for any $n \in \mathbb{Z}_{\geq 0}$ there exist $v_1, v_2 \in \mathcal{F}^*$ such that $d_{\mathcal{F}^*}(v_1, v_2) \geq n$.
- Let $T \in \pi^{-1}(v_1)$ and $T' \in \pi^{-1}(v_2)$
- We see that $n \leq d_{\mathcal{F}^*}(v_1, v_2) \leq d_{\mathcal{TP}}(T, T')$
- Thus, $d_{\mathcal{TP}}(T, T') \geq n$, and it follows that $\mathcal{TP}$ has an infinite diameter.

$\square$

Further Questions

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Thank you!