UNIVALENT APPROXIMATION BY FOURIER SERIES OF STEP FUNCTIONS

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ABSTRACT. We prove that univalent harmonic mappings can be approximated by univalent Fourier series of step functions.

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1. INTRODUCTION

The Fourier series of step functions arise in substantial ways both with regard to extremal properties of univalent harmonic mappings (cf. [4, pp, 59-72]), and in connection with minimal surfaces [8], [2]. A detailed study of the Fourier series of step functions was made by Sheil-Small in [11]. In this note we shall provide further information regarding these mappings.

In addition to the usual notation S_H and S_H^o used for univalent harmonic mappings and normalized univalent harmonic mappings in the unit disk U as in [3], we shall use:

$B(n)$ to denote the Poisson integrals of step functions f on ∂U with at most n steps, which map univalently onto positively oriented Jordan polygons,

B_n to denote those $f \in B(n)$ and normalized by $f(0) = 0$, $f_z(0) = 1$,

$\overline{B_n}$ to denote the closure of B_n with the topology of uniform convergence on compact subsets,

B_n^o to denote those $f \in B_n$ with $f_{\overline{z}}(0) = 0$,

$\overline{B_n^o}$ to denote the closure of B_n^o with the topology of uniform convergence on compact subsets.

Note that in analogy with [3, p.7], $\overline{B_n}$ consists of all functions

$$f = f_o + c\overline{f_o} \quad (f_o \in B_n^o, \quad |c| \leq 1).$$

2. THE APPROXIMATION THEOREM

Theorem 1. *If $F \in S_H^o$, then F can be approximated uniformly on compact subsets by functions in B_n^o.*

It may seem that Theorem 1 should be proved simply by interpolating $F(tz)/t$ by Poisson integrals of step functions. However, the task of making such approximations univalent in U seems daunting. Therefore, we shall use the tools involving first order elliptic systems as developed in [1] and [10].

Consider a first order system

$$u_x = a_{11}(x,y)v_x + a_{12}(x,y)v_y, \quad -u_y = a_{21}(x,y)v_x + a_{22}(x,y)v_y. \tag{2.1}$$

By a theorem of Bers and Nirenberg [1, p.132] we have

Theorem A. *If the coefficients of the system (2.1) are defined and continuous in a Jordan domain D and if the coefficients satisfy the ellipticity conditions*

$$4a_{12}a_{21} - (a_{11} + a_{22})^2 > 0 \quad \text{and} \quad a_{12} > 0, \tag{2.2}$$

then, given a Jordan domain Λ, there exists a solution $w = u + iv$ which is a homeomorphism of the closure of D onto the closure of Λ and takes three given boundary points of D onto three assigned boundary points on the boundary of Λ.

This will be supplemented with a result of Mcleod, Gergen, and Dressel [10, p. 174].

Theorem B. *Let D be bounded by a continuously differentiable Jordan curve and suppose that the coefficients a_{ij} in (2.1) are continuously differentiable in D and continuous on $\overline{D}$. Assume the system satisfies the ellipticity conditions (2.2), and let u_1, v_1 and u_2, v_2 be solution pairs of the system. If $F_1 = u_1 + iv_1$ and $F_2 = u_2 + iv_2$ are both continuously differentiable on D and continuous on $\overline{D}$, mapping D homeomorphically onto a set T such that three distinct points on ∂D correspond to the same three points on ∂T, then $F_1 \equiv F_2$ in D.*

We shall apply Theorem A and Theorem B to univalent harmonic mappings in U which satisfy

$$\overline{f_{\overline{z}}(z)} = a(z) f_z(z), \tag{2.3}$$

where the dilatation $a(z)$ is analytic in U and satisfies the condition $|a(z)| \le k < 1$.

Writing $f = u + iv$ and $a = a_1 + ia_2$ with $|a(z)| \le k < 1$ in U, (2.3) becomes

$$u_x - iv_x - i(u_y - iv_y) = (a_1 + ia_2)(u_x + iv_x - i(u_y + iv_y))$$

so that

$$\begin{aligned} u_x - v_y &= a_1(u_x + v_y) - a_2(v_x - u_y) \\ -u_y - v_x &= a_1(v_x - u_y) + a_2(u_x + v_y)). \end{aligned}$$

Solving these equations we obtain

$$\begin{aligned} u_x &= \frac{2a_2}{-a_2^2 - (1-a_1)^2} v_x + \frac{a_1^2 - 1 + a_2^2}{-a_2^2 - (1-a_1)^2} v_y \\ -u_y &= \frac{-a_2^2 + 1 - a_1^2}{a_2^2 + (1-a_1)^2} v_x + \frac{2a_2}{a_2^2 + (1-a_1)^2} v_y, \end{aligned}$$

and thus the conditions (2.2) are satisfied.

Our procedure will also rely on the work of Hengartner and Schober; in particular [7, Theorem 4.3].

Theorem C. *Let D be a bounded simply connected domain whose boundary is locally connected. Suppose that f is a univalent harmonic orientation preserving mapping from U into D for which the radial limits $\hat{f}(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ belong to ∂D for almost every θ. Then there exists a countable set $E \subseteq \partial U$ such that*
(a) the unrestricted limit $\hat{f}(e^{i\theta}) = \lim\limits_{z\to e^{i\theta}\, z\in U} f(z)$ exists, is continuous, and belongs to ∂D for $e^{i\theta} \in \partial U \backslash E$,
(b) $\lim_{\theta\uparrow\theta_0} \hat{f}(e^{i\theta})$ and $\lim_{\theta\downarrow\theta_0} \hat{f}(e^{i\theta})$ exist, are different, and belong to ∂D for $\varepsilon^{i\theta} \in E$,
(c) the cluster set of f at $e^{i\theta} \in E$ is a straight line segment joining $\lim_{\theta\uparrow\theta_0} \hat{f}(e^{i\theta})$ and $\lim_{\theta\downarrow\theta_0} \hat{f}(e^{i\theta})$.

Proof of Theorem 1. With F as in Theorem 1 having dilatation $A(z)$ and $0 < t < 1$, let $f(z) = f_t(z) = F(tz)/t$ so that $f(z) \in S_H^o$ having dilatation $a(z) = a_t(z) = A(tz) \le k$ for some $k < 1$. As such, the image $\Omega = \Omega_t = f(U)$ is a bounded domain with real analytic boundary. It suffices to approximate a fixed such $f(z)$ by taking t close to 1.

We fix three points $z_1,\ z_2,\ z_3$ positively oriented on ∂U, and the corresponding points $w_j = f(z_j)\ j = 1, 2, 3$ on $\partial\Omega$.

Since $\partial\Omega$ is smooth, we may take a sequence of Jordan polygons P_n interior to Ω having vertices $W_n = \{w_{n,1}, ...w_{n,k_n}\}$ such that $P_1 \subset P_2 \subset ....$, and having vertices with $w_1,\ w_2,\ w_3$ in such a way that $P_n \to \Omega$ in the sense that $\operatorname{dist}(\partial\Omega, \partial P_n) \to 0$, $\operatorname{dist}(w_{n,j}, w_{n,j+1}) \to 0$ $(j = 1, ..., , k_n,\ w_{n,k_n+1} = w_{n,1})$ uniformly as $n \to \infty$, and the lengths of the ∂P_n are uniformly bounded. We note that in the continuation, the term "vertices' can also refer to interior points of the line segments of the polygon.

We next take a sequence of Blaschke products $a_n(z)$ converging uniformly on compact subsets of U (in fact pointwise in U) to $a(z)$ [6, p. 7]. If $a_{n,\rho}(z) = a_n(\rho z)$ $(0 < \rho < 1)$, then by Theorem A, there exists a homeomorphism $f_{n,\rho}(z)$ of U onto P_n with $f_{n,\rho}(z_j) = w_j,\ j = 1, 2, 3$ and satisfying (2.3) with dilatation $a_{n,\rho}$.

In fact, since each $a_{n,\rho}$ satisfies $|a_{n,\rho}(z)| < k_{n,\rho} < 1$ for some constants $k_{n,\rho}$ and all $z \in U$, the $f_{n,\rho}$ in (2.3) are univalent harmonic mappings (cf. [4, p.6]).

Letting $\rho \to 1$, we can take subsequence which converges to a univalent harmonic mapping f_n of U into P_n, having dilatation $a_n(z)$. In fact the functions $f_{n,\rho}(z)$ are Poisson integrals of some boundary functions $\varphi_{n,\rho}(e^{i\theta})$ of uniformly bounded variation so that a subsequence converges to a function $\varphi_n(e^{i\theta})$ a.e., [5, p.3] and the limit function f_n is also a Poisson integral of a radial limit function $\psi_n(e^{i\theta})$. Thus $\psi_n(e^{i\theta}) = \varphi_n(e^{i\theta})$ a.e., and consequently $f_n(z)$ has radial limits on ∂P_n, a.e., and Theorem C applies.

It is important to emphasize that the functions f_n are in $B(n)$ since the dilatations are Blaschke products and there can be no nonconstant intervals of continuity on ∂U, since otherwise there would be an interval which is mapped onto a line segment which is not possible since the image of such an interval has to be strictly concave with respect to the interior (cf. [4, p. 116]. The 3 specified boundary points need not correspond. They either reside on the images of arcs where f_n is constant, or points of discontinuity which create the "collapsing line segments."

Again, the functions $f_n(z)$ are Poisson integrals of sense preserving step functions φ_n that have their values in the P_n respectively. We now take a subsequence of the f_n converging to a function $\tilde{f}$ thus having dilatation $a(z)$ Let $\tilde{f}_0$ denote the a.e. radial limit function for $\tilde{f}$. Then,

$$\tilde{f}(z) = \frac{1}{2\pi}\int_U P(r, \theta - t)\tilde{f}_0(e^{it})dt. \tag{2.4}$$

Since the functions $\{\varphi_n\}$ are of uniformly bounded variation, as before there exists a function φ on ∂U and a subsequence $\{\varphi_{n_k}\}$ converging a.e. to φ. Therefore, $\varphi(e^{i\theta}) = \tilde{f}_0(e^{i\theta})$ a.e., so that in particular, the values taken by $\tilde{f}_0(e^{i\theta})$ are a.e. in $\partial\Omega$.

Now $\tilde{f}(z)$ in (2.4) satisfies the conditions of Theorem C. Since $\tilde{f}$ has dilatation bounded strictly less than 1 it is quasiconformal in U and hence a homeomorphism on $\overline{U}$ [9, p.98]. This implies that the set E in Theorem C is empty, and thus it follows that $\hat{f}$ must map ∂U homeomorphically onto ∂D. From our construction it follows further that $\hat{f}(z_j) = w_j \quad j = 1, 2, 3$. Thus, from Theorem B we conclude that $\tilde{f}(z) \equiv f(z)$.

The functions $\{f_n\}$ that approximate f are in $B(n)$. Set $f_n(z) = \sum_{k=0}^{\infty} a_{nk}z^k + \sum_{k=1}^{\infty} \overline{b_{nk}z^k}$. Since the f_n converge locally uniformly to $f(z)$, we infer that a_{n0}, a_{n1}, b_{n1} converge to $0, 1, 0$ respectively. Accordingly the functions

$$g_n(z) = \frac{\overline{a}_{n1}(f_n(z) - a_{n0}) - \overline{b}_{n1}\overline{(f_n(z) - a_0^n)}}{|a_1^n|^2 - |b_1^n|^2} \in B_n^0$$

are univalent harmonic mappings converging locally uniformly to $f(z)$. This completes the proof. □

3. Growth of functions in B_n^o

In B_n^o the h' and g' are rational functions of order at most n, with poles of order 1. In $\overline{B_n^o}$ the h' and g' are still rational functions of order at most n, but in the closure the poles may coalesce to create poles of higher order. If ζ_k is such a point, then locally the corresponding terms in the series are of the form $(z - \zeta_k)^{-m_k}P(z - \zeta_k)$ for h' and $(z - \zeta_k)^{-m_k}Q(z - \zeta_k)$ where P and Q are polynomials. Since $g'(z) = a(z)h'(z)$ where $a(z)$ is a finite Blaschke product, it follows that $|P(\zeta_k)| = |Q(\zeta_k)|$.

Theorem 2. *If $f = h + g \in \overline{B_n^o}$ and h has a pole at $\zeta \in \partial U$, then the order of the pole is at most 3.*

Proof of Theorem 2. Arguing by contradiction, we assume that h has a pole of order k at least 4, and we consider only even k. The odd case is similar. We may assume that $\zeta = 1$.

As described above, we then have

$$w = f(z) = \frac{e^{i\alpha}}{(z-1)^k} + \frac{e^{i\beta}}{(\overline{z}-1)^k} + \text{lower order terms}$$

$$= e^{i(\alpha+\beta)/2}\left(\frac{e^{i(\alpha-\beta)/2}}{(z-1)^k} + \frac{e^{i(-\alpha+\beta)/2}}{(\overline{z}-1)^k}\right) + \text{lower order terms}$$

$$= 2e^{i(\alpha+\beta)/2}\Re e\frac{e^{i(\alpha-\beta)/2}}{(z-1)^k} + \text{lower order terms.}$$

Writing $z - 1 = re^{i\varphi}$ and $(\alpha-\beta)/2 = \varphi_0,\ (\alpha+\beta)/2 = \varphi_1$ we have

$$w = f(z) = 2e^{i\varphi_1}\frac{\cos k(\varphi - \varphi_0/k)}{r^k} + \text{lower order terms.} \tag{3.1}$$

By a rotation we may ignore the term $e^{i\varphi_1}$ in (3.1).

We require some notation. Let $\varepsilon > 0$, and $0 < \delta < 1$. Let $\Delta = \Delta(\varepsilon, \delta)$ be the portion of U between $|z-1| = \varepsilon$ and $|z-1| = \varepsilon^\delta$. The boundary of Δ is a simple closed curve and, for small ε, φ ranges on an interval only slightly smaller than $(\pi/2, 3\pi/2)$. Therefore, since $k \geq 4$, on the side where $|z-1| = \varepsilon$ there will be at least 3 consecutive intervals of the form $\alpha < k(\varphi - \varphi_0/k) < \alpha + \pi$. Let I_0 be the middle one of a set of 3 consecutive intervals, and

$$W = \{w : |\Re e\, w| < \varepsilon^{-(1+\delta)/2}.$$

For small ε, a portion of $f(I_0)$ extends outside of W and the rest inside.. We assume that it is on the right side; the proof for the left side would be similar.

As φ increases, the portion of $f(I_0)$ outside W has an initial value $x+iy_1$ and terminal value $x + iy_2$. Since $f(\partial\Delta)$ is sense preserving, it must be that $y_1 > y_2$. Regarding the images of the two intervals adjacent to I_0, again because of the sense preserving nature of $f(\partial\Delta)$, the portions of their images as they exit and reenter W on the left side must turn towards each other. This means that there can be no accommodation for another portion of the image of $f(\partial\Delta)$ to exit again on the right side without crossing. Thus it cannot be that $k \geq 4$.. □

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