Low Tensor-Train Rank Methods to Solve Sylvester Tensor Equations

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Sylvester matrix equation

$$AX + XB^T = F$$

$$A \in \mathbb{C}^{n_1 \times n_1}, \quad B \in \mathbb{C}^{n_2 \times n_2}, \quad F \in \mathbb{C}^{n_1 \times n_2}$$

3D Sylvester tensor equation

$$\mathcal{X} \times_1 A + \mathcal{X} \times_2 B + \mathcal{X} \times_3 C = \mathcal{F}$$

$$A \in \mathbb{C}^{n_1 \times n_1}, \quad B \in \mathbb{C}^{n_2 \times n_2}, \quad C \in \mathbb{C}^{n_3 \times n_3}, \quad \mathcal{F} \in \mathbb{C}^{n_1 \times n_2 \times n_3}$$

k-mode product for a tensor $\mathcal{X} \in \mathbb{C}^{n_1 \times \cdots \times n_d}$ and a matrix $A \in \mathbb{C}^{n_k \times n_k}$

$$(\mathcal{X} \times_k A)_{i_1,\ldots,i_{k-1},j,i_{k+1},\ldots,i_d} = \sum_{i_k=1}^{n_k} \mathcal{X}_{i_1,\ldots,i_d} A_{j,i_k}$$

Example: Poisson w/ FD

$$-(u_{xx} + u_{yy} + u_{zz}) = f \text{ on } \Omega = [-1,1]^3$$

$$u|_{\partial\Omega} = 0$$

$$\mathcal{X} \times_1 K + \mathcal{X} \times_2 K + \mathcal{X} \times_3 K = \mathcal{F}$$

$$K = -\frac{1}{h^2} \begin{bmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix}$$

Alternating Direction Implicit (ADI) method

[Wachspress, 2008]

"Iterative" method for Sylvester matrix equation $AX - XB^T = F$

1. Select shift parameters p and q each of length ℓ based on spectra of A and B.
2. For $1 \leq j \leq \ell$
   - Solve $(A - q_j I)X = F + X(B - q_j I)^T$.
   - Solve $X(B - p_j I)^T = (A - p_j I)X - F$.

Factored ADI (fADI) [Benner, Li & Truhar, 2009]

"Iterative" method for Sylvester matrix equation $AX - XB^T = UV^T$

$X = ZDY^T$ rank r

1. Select shift parameters p and q each of length ℓ based on spectra of A and B.
2. Solve $(A - q_1 I)Z_1 = U$ and $(B - \bar{p}_1 I)Y_1 = V$. Set $Z = Z_1$ and $Y = Y_1$.
3. Let $D = (q_1 - p_1)I$.
4. For $1 \le j \le \ell - 1$
   - Solve $(A - q_{j+1}I)Z_{j+1} = (q_{j+1} - p_j)Z_j$. Set $Z_{j+1} = Z_{j+1} + Z_j$ and $Z = \begin{bmatrix} Z & Z_{j+1} \end{bmatrix}$.
   - Solve $(B - \bar{p}_{j+1}I)Y_{j+1} = (\bar{p}_{j+1} - \bar{q}_j)Y_j$. Set $Y_{j+1} = Y_{j+1} + Y_j$ and $Y = \begin{bmatrix} Y & Y_{j+1} \end{bmatrix}$.
   - Set $D = \begin{bmatrix} D & \\ & (q_{j+1} - p_{j+1})I \end{bmatrix}$

fADI as a direct method

- Guaranteed to converge after all iterations
- Quasi-optimal shift parameters p and q are known in many situations

[Fortunato & Townsend, 2020]
[Townsend & Wilber, 2018]

- Zeros and poles of a rational function that can achieve a quasi-optimal Zolotarev number [Zolotarev, 1877]

$$Z_k(\Lambda(A), \Lambda(B)) := \inf_{r \in \mathscr{R}_{k,k}} \frac{\sup_{z \in \Lambda(A)} |r(z)|}{\inf_{z \in \Lambda(B)} |r(z)|}, \qquad k \geq 0$$

$$||X - X_k||_F \leq Z_k(\Lambda(A), \Lambda(B)) ||X||_F$$

[S. & Townsend, 2021]

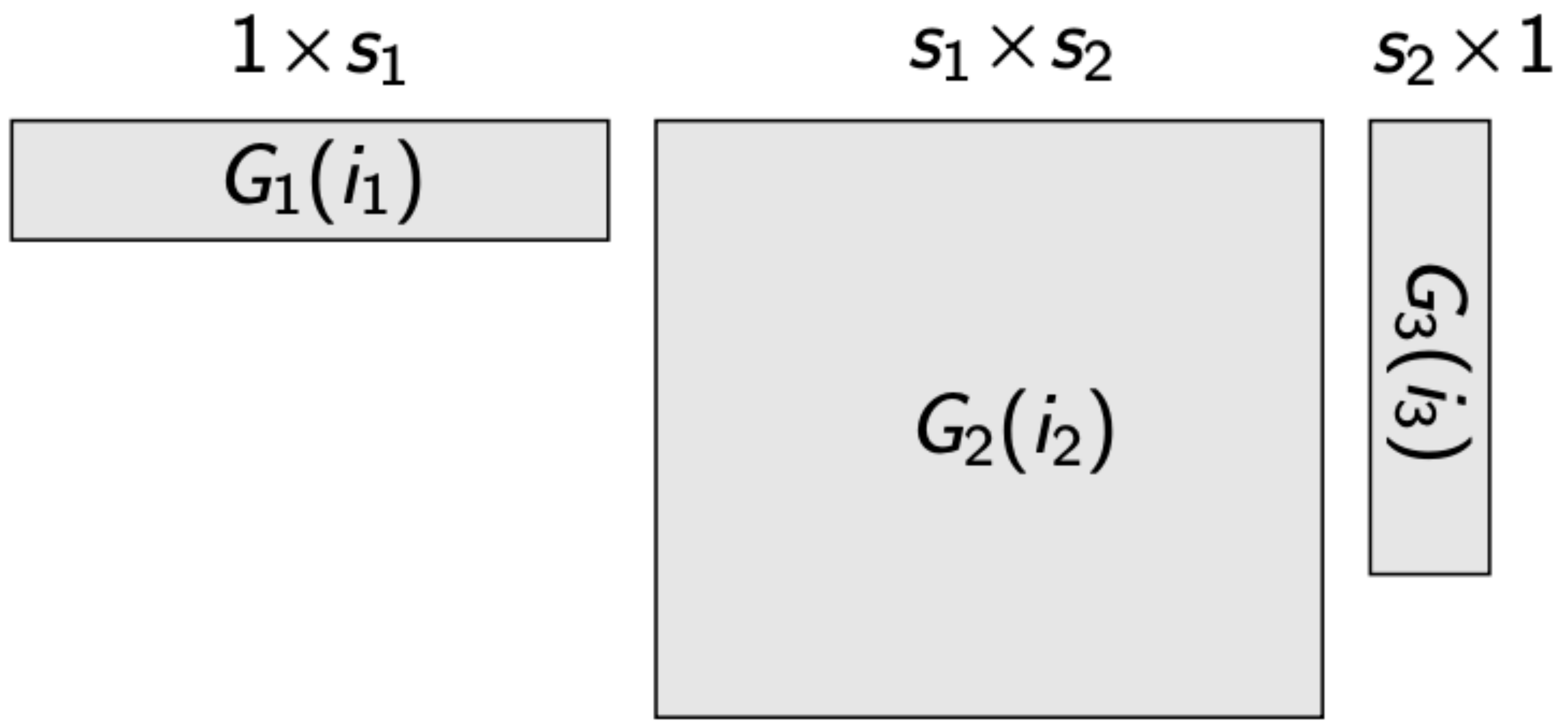
- Complexity $\mathcal{O}(r\ell T)$, $\mathcal{O}(T)$ for solving shifted linear systems of A and B

Goal: Solve Sylvester tensor equations with fADI

Method: Rewrite into several Sylvester matrix equations

Tensor-train format [Oseledets, 11]

$\mathcal{X}_{i_1,i_2,i_3} \approx$



$\mathcal{G}_j \in \mathbb{C}^{s_{j-1}\times n_j \times s_j}$

$s_k \leq \text{rank}(X_k), \qquad X_k = \text{reshape}(\mathcal{X}, \prod_{s=1}^{k} n_s, \prod_{s=k+1}^{3} n_s)$

Trains: “somewhat” mimic the bases for column/row spaces of a matrix

$$\mathcal{X} \times_1 A + \mathcal{X} \times_2 B + \mathcal{X} \times_3 C = \mathcal{F}$$

$F_1 = M_1 N_1^T$ and $F_2 = M_2 N_2^T$ ⟷ $\mathcal{F}$ with TT cores $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ have low TT rank

$F_1 = G_1 (G_2)_1 (G_3 \otimes I) \quad F_2 = (I \otimes G_1)(G_2)_2 G_3$

3D Sylvester solver in TT format

$$\mathscr{X} \times_1 A + \mathscr{X} \times_2 B + \mathscr{X} \times_3 C = \mathscr{F}$$

Method 1: Combine TTSVD with fADI

TTSVD [Oseledets, 11]

Calculate SVD of $X_1 \approx U_1 S_1 V_1^T$,
and use U_1 as first "train"

Let $W = \mathrm{reshape}(S_1 V_1^T, s_1 n_2, n_3)$,
and calculate SVD of $W \approx U_2 S_2 V_2^T$.
Use $\mathrm{reshape}(U_2, s_1, n_2, s_2)$ as
second "train", and $S_2 V_2^T$ as third
"train"

fADI steps [S. & Townsend, 21]

$$AX_1 + X_1(I \otimes B + C \otimes I)^T = F_1 = M_1 N_1^T$$

Solve only for column space basis U_1

$$(I \otimes (U_1^T A U_1) + B \otimes I)W + WC^T = (I \otimes U_1^T)F_2 = (I \otimes U_1^T)M_2 N_2^T$$

$$(I \otimes (U_1^T A U_1) + B \otimes I - \alpha I)Z = R$$

$$\updownarrow$$

$$(U_1^T A U_1 - \frac{\alpha}{2} I)Z_j + Z_j(B - \frac{\alpha}{2} I)^T = R_j$$

Z_j and R_j are reshape of jth column of Z and R

Example: Poisson equation

$$-(u_{xx}+u_{yy}+u_{zz}) = f \text{ on } \Omega = [-1,1]^3, \qquad u|_{\partial\Omega} = 0$$

Ansatz $\tilde{C}^{(3/2)}$: normalized ultraspherical

$$u(x,y,z) = (1-x^2)(1-y^2)(1-z^2)\sum_{p=0}^{n}\sum_{q=0}^{n}\sum_{r=0}^{n}\mathscr{X}_{pqr}\tilde{C}_p^{(3/2)}(x)\tilde{C}_q^{(3/2)}(y)\tilde{C}_r^{(3/2)}(z),$$

$$f(x,y,z) = \sum_{p=0}^{n}\sum_{q=0}^{n}\sum_{r=0}^{n}\mathscr{F}_{pqr}\tilde{C}_p^{(3/2)}(x)\tilde{C}_q^{(3/2)}(y)\tilde{C}_r^{(3/2)}(z)$$

Sylvester equation

$$\mathscr{X}\times_1 A^{-1} + \mathscr{X}\times_2 A^{-1} + \mathscr{X}\times_3 A^{-1} = \mathscr{G}$$

$$\mathscr{G} = \mathscr{F}\times_1 M^{-1}\times_2 M^{-1}\times_3 M^{-1}$$

$A = D^{-1}M$, D diagonal, M and A symmetric pentadiagonal

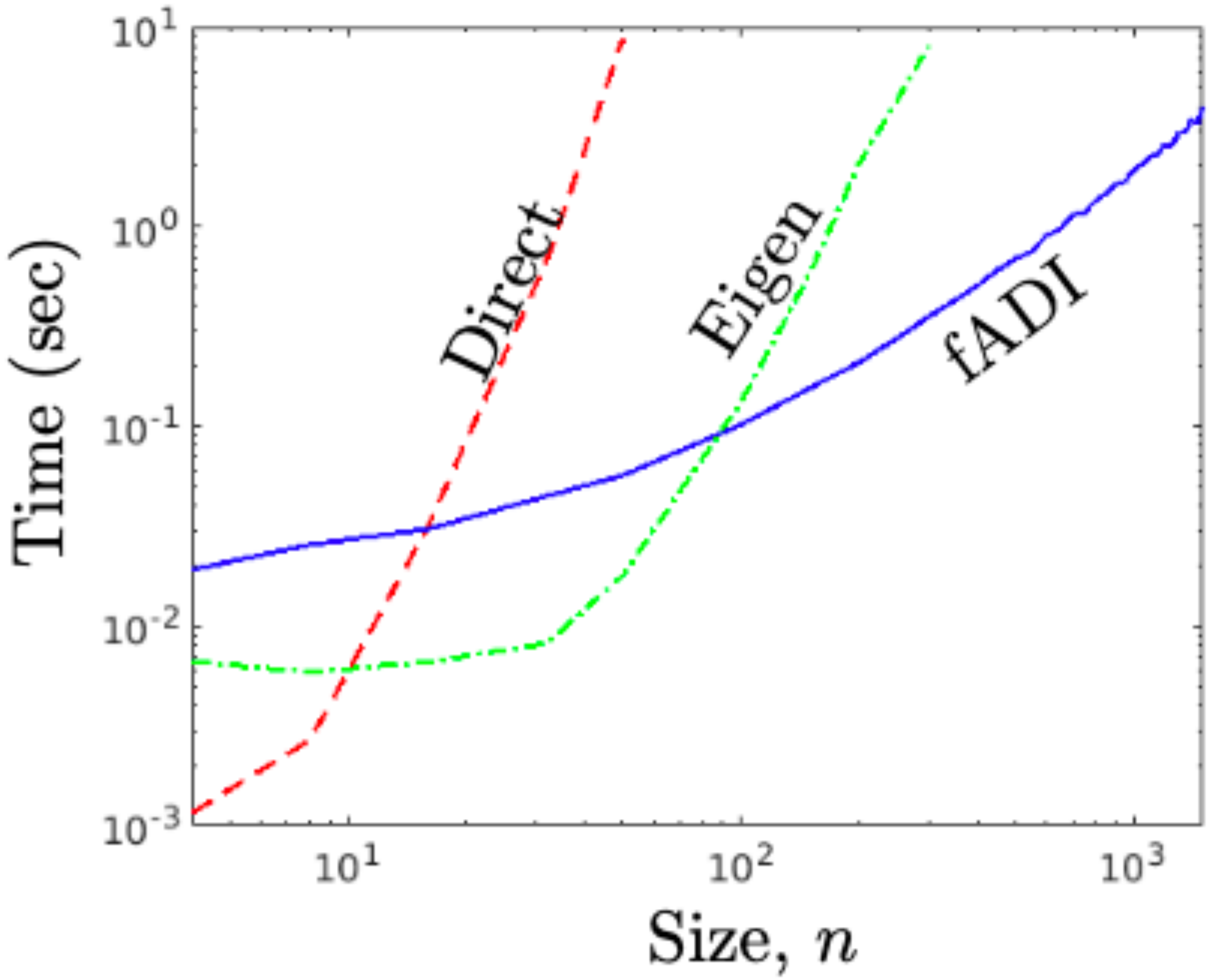
$$\Lambda(A) \in [-1, \ -1/(30n^4)]$$

[Fortunato & Townsend, 20]

$$f = -2(1-y^2)(1-z^2) - 2(1-x^2)(1-z^2) - 2(1-x^2)(1-y^2)$$

$$u = (x^2-1)(y^2-1)(z^2-1)$$

Complexity $\mathcal{O}(n(\log n)^3(\log(1/\epsilon)^3)$



3D Sylvester solver in TT format

$$\mathscr{X} \times_1 A + \mathscr{X} \times_2 B + \mathscr{X} \times_3 C = \mathscr{F}$$

Method 2: Combine parallel-TTSVD with fADI

parallel-TTSVD [S., Ruth & Townsend]

Calculate SVD of $X_1 \approx U_1 S_1 V_1^T$,
and use U_1 as first "train"

Calculate SVD of $X_2 \approx U_2 S_2 V_2^T$. Use
$\text{reshape}(U_1^*\text{reshape}(U_2, n_1, n_2 s_2), s_1, n_2, s_2)$
as second "train" and $S_2 V_2^T$ as third "train"

fADI steps [S., Ruth & Townsend]

$$AX_1 + X_1(I \otimes B + C \otimes I)^T = F_1 = M_1 N_1^T$$

Solve only for column space basis U_1

$$(I \otimes A + B \otimes I)X_2 + X_2 C^T = F_2 = M_2 N_2^T$$

Only solve shifted linear systems with A, B, and C

Use universal shift parameters for both above equations, then find "trains" almost simultaneously

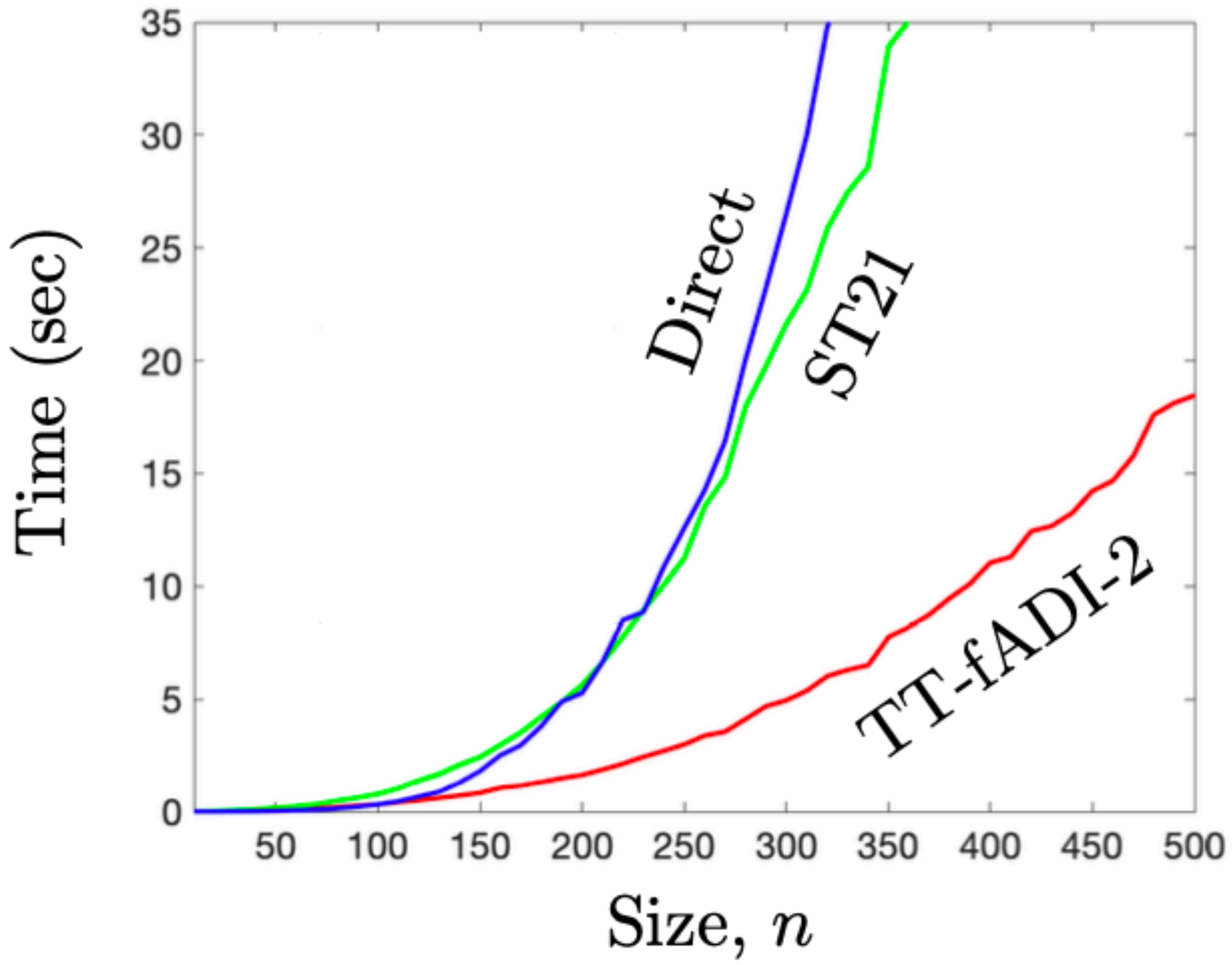
Example

$$\mathscr{X} \times_1 D + \mathscr{X} \times_2 D + \mathscr{X} \times_3 D = \mathscr{F}$$

$D \in \mathbb{R}^{n\times n}$ diagonal with $D_j \in [-1, -1/(30n)]$

$\mathscr{F}$ has TT rank $(1, \lfloor n/4 \rfloor, 2, 1)$ with i.i.d. uniform random numbers in TT cores

Complexity $\mathcal{O}(n(\log n)^3(\log(1/\epsilon)^3)$



Thank you!