EXCEPTIONAL COLLECTION OF OBJECTS ON SOME FAKE PROJECTIVE PLANES

CHING-JUI LAI, SAI-KEE YEUNG

ABSTRACT. The purpose of the article is to explain a new method to study existence of a sequence of exceptional collection of length three for fake projective planes M with large automorphism group. This provides more examples to a question in [GKMS].

1. Introduction

1.1 A fake projective plane is a compact complex surface with the same Betti numbers as $P^2_{\mathbb{C}}$. This is a notion introduced by Mumford who also constructed the first example. All fake projective planes have recently been classified into twenty-eight non-empty classes by the work of Prasad-Yeung in [PY], which finally leads to 100 fake projective planes along those 28 classes in the work of Cartwright-Steger [CT]. It is known that a fake projective plane is a smooth complex two ball quotient, and has the smallest Euler number among smooth surfaces of general type.

Most of the fake projective planes have the property that the canonical line bundle K_M can be written as $K_M = 3L$, where L is a generator of the Neron-Severi group, see Lemma 1 for the complete list. One motivation of the present article comes from a question of Dolgachev and Prasad, who asked whether $\mathrm{H}^0(M, 2L)$ contains enough sections for geometric purposes, such as embedding of M. It is also questioned in [GKMS] whether $\mathrm{H}^0(M, 2L)$ is non-trivial.

The other motivation comes from the recent research activities surrounding the search of exceptional collections from the point of view of derived category, such as [AO], [BvBS], [F], [GS], [GKMS] and [GO].

1.2 Denote by $D^b(M)$ the bounded derived category of coherent sheaves on M. A sequence of objects $E_1, E_2, ..., E_r$ of $D^b(M)$ is called an exceptional collection if $\mathrm{Hom}(E_j, E_i[k])$ is non-zero for $j \geq i$ and $k \in \mathbb{Z}$ only when $i = j$ and $k = 0$, in which case it is one dimensional. In [GKMS], the authors consider the problem of the existence of a special type exceptional collection on an n-dimensional fake projective space.

Conjecture 1([GKMS]). *Assume that M is an n-dimensional fake projective space with the canonical class divisible by $(n+1)$. Then for some choice of $\mathcal{O}_M(1)$ such*

Key words: Fake projective planes. Exceptional collection.
AMS 2010 Mathematics subject classification: Primary 14J29, 13D09.
The first author was partially supported by a grant from the National Science Foundation.

that $\omega_M = \mathcal{O}_M(n+1)$, the sequence

$$\mathcal{O}_M, \mathcal{O}_M(-1), \ldots, \mathcal{O}_M(-n)$$

is an exceptional collection on M.

In the cases of fake projective planes ($n = 2$), it is easy to see that a necessary and sufficient condition for the above conjecture is to show that $\mathrm{H}^0(M, 2L) = 0$. This is proved in [GKMS] if $\mathrm{Aut}(M)$ has order 21. This is also proved for 2-adically uniformised fake projective planes in [F]. The main result in this note aims to provide more examples to the conjecture of [GKMS]. Since our method depends mostly on the numerical property, we propose the following slightly more general problem which seems to be more accessible and still serves the purpose of providing exceptional objects.

Conjecture 2. *Assume that M is an n-dimensional fake projective space with the canonical class* ***numerically*** *divisible by $(n+1)$. Then for some choice of L such that $K_M \equiv (n+1)L$ and a suitable choices of line bundles E_i's with $E_i \equiv -iL$, $1 \leq i \leq n$, the sequence*

$$\mathcal{O}_M, E_1, E_2, \ldots, E_n$$

is an exceptional collection on M.

1.3 The problem is rather subtle, since the conventional Riemann-Roch formula is not useful in this case without an appropriate vanishing theorem. The approach that we take exploits the small intersection numbers involved as well as existence of a finite group action.

We illustrate our approach by proving the above conjecture for fake projective planes with $\mathrm{Aut}(X) = C_3 \times C_3$ and $C_7 : C_3$, the latter case was proved in [GKMS]. The approach is geometric and is different from [GKMS] and [F]. We choose L to be an $\mathrm{Aut}(X)$-invariant numerical cubic root of K_M. The problem is reduced to a study of the geometry of invariant sections of $\mathrm{H}^0(M, 2L)$ if it exists.

Main Theorem. *For M a fake projective plane as listed in the Table below, there is a unique line bundle L with $K_M = 3L$. Moreover, the sequence $\mathcal{O}_M, -L, -2L$ forms an exceptional collection of M.*

class	M	$\mathrm{Aut}(M)$	$\mathrm{H}_1(M, \mathbb{Z})$
$(a = 7, p = 2, \emptyset)$	$(a = 7, p = 2, \emptyset, D_3, 2_7)$	$C_7 : C_3$	C_2^4
$(a = 7, p = 2, \{7\})$	$(a = 7, p = 2, \{7\}, D_3, 2_7)$	$C_7 : C_3$	C_2^3
$(\mathcal{C}_2, p = 2, \emptyset)$	$(\mathcal{C}_2, p = 2, \emptyset, d_3, D_3)$	$C_3 \times C_3$	$C_2 \times C_7$
$(\mathcal{C}_2, p = 2, \{3\})$	$(\mathcal{C}_2, p = 2, \{3\}, d_3, D_3)$	$C_3 \times C_3$	C_7
$(\mathcal{C}_{18}, p = 3, \emptyset)$	$(\mathcal{C}_{18}, p = 3, \emptyset, d_3, D_3)$	$C_3 \times C_3$	$C_2^2 \times C_{13}$
$(\mathcal{C}_{20}, \{v_2\}, \emptyset)$	$(\mathcal{C}_{20}, \{v_2\}, \emptyset, D_3, 2_7)$	$C_7 : C_3$	C_2^6

We remark that the above table covers 12 different fake projective planes up to biholomorphism. As mentioned earlier, the results for $\mathrm{Aut}(M) = C_7 : C_3$ have been obtained earlier in [GKMS] by a different method.

Related to Conjecture 2, our method of proof implies immediately the following slightly stronger result, since only numerical properties of the line bundles are involved in the proof.

Main Theorem'. *Let M be a fake projective planes described in the Main Theorem. Let E_1 and E_2 be any $Aut(M)$-invariant torsion line bundles on M. Let $L_i = L+E_i$, $i = 1, 2$. Then $\mathcal{O}_M, -L_1, -2L_2$ forms an exceptional collection of M.*

After the completion of the first draft of the paper, the main results of this paper were presented at the 4th South Kyushu workshop on algebra, complex ball quotients and related topics, July 22-25, 2014, Kumamoto, Japan. The second author thanks Fumiharu Kato for his kind invitation. During the conference, J. Keum mentioned that he had obtained similar results as well.

2. Line bundles on fake projective planes

2.1 We work over $\mathbb{C}$. Throughout this paper, we denote by C_m the cyclic group of order m, $\sim$ the linear equivalence, and $\equiv$ the numerical equivalence. Also, we denote by $C_7 : C_3$ the unique (up to isomorphism) nonabelian finite group of order 21,

$$C_7 : C_3 = \langle x, y | x^3 = y^3 = 1, xyx^{-1} = y^2 \rangle.$$

2.2. Let M be a fake projective plane. It follows from definition of M and Poincaré Duality that K_M is equal to $3L$ modulo torsion for some line bundle L which can be taken to be a generator of the torsion-free part of the Neron-Severi group. First of all, we would like to list all fake projective planes with $K_M = 3L$.

Lemma 1. *Among the* 100 *fake projective planes,* 92 *of which satisfies the property that $K_M = 3L$.*

Proof. Recall that a fake projective plane is a complex two ball quotient $B^2_{\mathbb{C}}/\Pi$ for an arithmetic group Π classified in [PY] and [CS]. From the argument of §10.2 of [PY], it is known that $K_M = 3L$ if and only if Γ can be lifted to become a lattice in $\mathrm{SU}(2, 1)$, and $K_M = 3L$ if the second cohomology class of M has no three torsion. The latter fact is an immediate consequence of the Universal Coefficient Theorem, see **2.3** below or Lemma 3.4 of [GKMS]. The section §10.2 of [PY] also shows that Γ can be lifted to $\mathrm{SU}(2, 1)$ if the number fields involved is not one of the types $\mathcal{C}_2$ or $\mathcal{C}_{18}$. There are 12 candidates for Π lying in $\mathcal{C}_2$ or $\mathcal{C}_{18}$. Out of these 12 examples, 3 of them do not have 3-torsion elements in $\mathrm{H}^2(M, \mathbb{Z})$ and hence the corresponding Π can be lifted to $\mathrm{SU}(2, 1)$. Finally, it is listed in the file registerofgps.txt of the weblink of [CS], that the lattices can be lifted to $\mathrm{SU}(2, 1)$ except for four cases in $\mathcal{C}_{18}$, corresponding to $(\mathcal{C}_{18}, p = 3, \{2\}, D_3), (\mathcal{C}_{18}, p = 3, \{2\}, (dD)_3), (\mathcal{C}_{18}, p = 3, \{2\}, (d^2D)_3)$ and $(\mathcal{C}_{18}, p = 3, \{2I\})$ in the notation of the file, see also Table 2 in [Y2]. Since there are two non-biholomorphic conjugate complex structures on such surfaces, it leads to the result that 92 of the fake projective planes can be regarded as quotient of $B^2_{\mathbb{C}}$ by a lattice in $\mathrm{SU}(2, 1)$.

□

2.3 For a smooth projective surface S, any holomorphic line bundle represents an element in the Neron-Severi group $i_*\mathrm{H}^2(S, \mathbb{Z}) \cap \mathrm{H}^{1,1}(S)$, where $i : \mathbb{Z} \to \mathbb{C}$ is the inclusion map. Let us consider the torsion part $\mathrm{H}^2(S,\mathbb{Z})$, which gives rise to torsion line bundles. From the Universal Coefficient Theorem, we have

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{H}_1(S,\mathbb{Z}),\mathbb{Z}) \to \mathrm{H}^2(S,\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_2(S,\mathbb{Z}),\mathbb{Z}) \to 0.$$

Since $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_2(S,\mathbb{Z}),\mathbb{Z}))$ is torsion free, for the sake of computation of torsion part of $i_*\mathrm{H}^2(S,\mathbb{Z}) \cap \mathrm{H}^{1,1}(M)$, it suffices for us to investigate $\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{H}_1(S,\mathbb{Z}),\mathbb{Z})$. On the other hand, for any abelian group A, we know that $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z},A) \cong A/mA$. Hence p-torsions of $\mathrm{H}^2(S,\mathbb{Z})$ corresponds to p-torsions of $\mathrm{H}_1(S,\mathbb{Z})$.

Lemma 2. *For a fake projective plane M with no 3 torsion in $\mathrm{H}_1(M,\mathbb{Z})$, we may assume that $K_M = 3L$ for an $\mathrm{Aut}(M)$-invariant line bundle L. In particular, the space of sections $\mathrm{H}^0(M,kL)$, if non-zero, is an $\mathrm{Aut}(M)$-module.*

Proof. This follows essentially from the discussions above. The first statement was observed in Lemma 3.4 of [GKMS], see also §10.3 and §10.4 of [PY]. From the above discussions, as there is no 3 torsion in $\mathrm{H}_1(M,\mathbb{Z})$ and hence in $\mathrm{H}^2(M,\mathbb{Z})$, we may write $E = 3E_1$ for a torsion line bundle E_1. We may simply let $L = L_1 + E_1$. Again, L is unique since there is no 3-torsion. Hence L is $\mathrm{Aut}(M)$-invariant. □

3. Holomorphic sections and group actions

3.1 From this point on, we assume that the automorphism group $\mathrm{Aut}(M)$ of M is non-trivial. We start with a simple statement, which has also been observed in [GKMS].

Lemma 3. *Let M be a fake projective space. Then $h^0(M,2L) \leqslant 2$.*

Proof. Consider the homomorphism

$$\mathrm{H}^0(M,2L) \times \mathrm{H}^0(M,2L) \xrightarrow{\alpha} \mathrm{H}^0(M,4L),$$

given by $\alpha(x,y) = x \times y$. This induces a mapping

$$\mathbb{P}(\mathrm{H}^0(M,2L)) \times \mathbb{P}(\mathrm{H}^0(M,2L)) \xrightarrow{\beta} \mathbb{P}(\mathrm{H}^0(M,4L))$$

The restriction of β to the first factor with a fixed value in the second factor is clearly injective. Similarly when we reverse the roles of the first and the second factor. Apply now a classical result of Remmert-Ven der Van [RV], p.155, it follows that the image of β has dimension the same as dimension of the domain. It follows that $h^0(M,4L) \geqslant 2(h^0(M,2L)-1)$. Since $h^0(M,4L) = 3$ by Riemann-Roch formula and Kodaira vanishing theorem, it follows that $h^0(M,2L) \leqslant 2$.

□

3.2. We study the action of $\mathrm{Aut}(M)$ on the section space $|2L|$ if it is nonempty.

Lemma 4. *Let M be a fake projective plane with $K_M = 3L$, where L is invariant under $\mathrm{Aut}(M)$. Suppose that $h^0(M,2L) \neq 0$, then for any non-trivial subgroup $H < \mathrm{Aut}(M)$, there exists a section $\sigma \in \mathrm{H}^0(M,2L)$ such that the divisor Σ associated to σ is invariant under H. Here H acts non-trivially on Σ.*

Moreover, if Σ is not an irreducible and reduced curve, then one of the following holds:

(1) $\Sigma = \Sigma_1 + \Sigma_2$*, where* Σ_i*'s are irreducible and reduced curves, and* Σ_1 *and* Σ_2 *intersects transversally at a smooth point.*

(2) $\Sigma = 2C$*, where* C *is an irreducible and reduced curve.*

Proof. If $h^0(M, 2L) = 1$, then there exists an effective divisor $\Sigma \sim 2L$ and is unique. Since $h^*L = L$, we conclude that $h^*\Sigma = \Sigma$, as asserted by the lemma.

Assume now that $h^0(M, 2L) = 2$. Let C_1, C_2 be two linearly independent sections of $\mathrm{H}^0(M, 2L)$. It follows that for all $a, b \in \mathbb{C}$, $aC_1 + bC_2$ is a global section of $\mathrm{H}^0(M, 2L)$ as well. Hence we get a complete linear system $aC_1 + bC_2$, with $[a : b] \in P^1_{\mathbb{C}}$. Under the action of $h \in H$, h^*C_i becomes a section of the form $aC_1 + bC_2$. Hence there is an induced action of H on $P^1_{\mathbb{C}}$. From Riemann-Hurwitz Formula, we know that $P^1_{\mathbb{C}}$ does not have non-trivial torsion free quotient. Hence there is at least one fixed point for the action of H. This corresponds to an effective divisor $\Sigma \sim 2L$ invariant under H.

We claim that H cannot act trivially on Σ. Assume on the contrary that it acts trivially on Σ. It follows that Σ is fixed pointwise by H. Since H is finite and Σ is complex dimension 1, we observe that Σ must be totally geodesic. To see this, consider a real geodesic curve $c(t), |t| < \epsilon$ on M with initial point $p \in \Sigma$ and initial tangent $\tau_p = c'(0) \in T_p\Sigma$. As both p and $c'(0)$ are fixed by H, the whole geodesic curve $c(t), |t| < \epsilon$ is fixed by H since the differential equation governing $c(t)$ is a second order ordinary equation and is determined by the initial conditions specified above. It follows that $c(t)$ actually lies on Σ. Since this is true for all smooth points $p \in \Sigma$ and $\tau_p \in T_p\Sigma$, we conclude that Σ is totally geodesic. On the other hand, from the result of [PY], we know that the arithmetic lattice Π associated to M is arithmetic of second type. It follows that there is no totally geodesic curve on M, cf. Lemma 8 of [Y2]. The claim is proved.

Suppose that Σ is not integral. If $\Sigma = \sum_i m_i\Sigma_i$, where Σ_i's are irreducible and reduced, then $\Sigma_i \equiv n_iL$ for some $n_i \in \mathbb{Z}_{>0}$ as $\rho(X) = 1$ and L is a generator of the Neron-Severi group. Since $\Sigma \equiv 2L$, $\sum_i m_in_i = 2$. Hence either $\Sigma = \Sigma_1 + \Sigma_2$, or $\Sigma = 2C$. Moreover, if $\Sigma = \Sigma_1 + \Sigma_2$, then $\Sigma_1.\Sigma_2 = 1$ and they must intersect transversally exactly at one smooth point.

□

3.3 Now we apply Lefschetz Fixed Point Theorem to analyze the geometry of an H-invariant curve Σ guaranteed in Lemma 4. As we will see in Lemma 7, this action always has fixed points. We will use the following lemma, cf. [P].

Lemma 5. *Let* C *be a compact Riemann surface. Let* $1 \neq g \in \mathrm{Aut}(C)$ *be an element of prime order* l *acting non-trivially on* C *with* n *fixed points. Denote* $(\ )^{\mathrm{inv}}$ *the eigenspace of* g *of eigenvalue* 1*. Then for* $\Delta = g(C) - dim_{\mathbb{C}}\mathrm{H}^1(\mathcal{O}_C)^{\mathrm{inv}}$*, we have*

$$n = 2 - 2g(C) + \frac{2l}{l-1}\Delta. \tag{1}$$

Proof. We consider the holomorphic Lefschetz fixed point theorem,

$$\sum_{gp=p} \frac{1}{\det(1 - \mathcal{J}_p(g^k))} = \mathrm{tr}((g^k)^*|_{\mathrm{H}^0(C,\mathcal{O}_C)}) - \mathrm{tr}((g^k)^*|_{\mathrm{H}^1(C,\mathcal{O}_C)}),$$

where $\mathcal{J}_p(g^k)$ is the holomorphic Jacobian with respect to the action of g^k at a fixed point p.

Consider summing up $k = 1, \ldots, l-1$ of the above formula. For the complex $\langle g \rangle$-module $V = \mathrm{H}^1(C, \mathcal{O}_C)$ of an element $1 \neq g \in \mathrm{GL}_{\mathbb{C}}(V)$ with prime order l, we have

$$\sum_{k=1}^{l-1} \mathrm{tr}((g^k)^*|_{\mathrm{H}^1(C,\mathcal{O}_C)}) = (l-1)(g(C) - \Delta) - \Delta = (l-1)g(C) - l\Delta.$$

For the left hand side of Lefschetz formula, since C is one-dimensional, $\mathcal{J}_p(g^k) = \rho^k$, where ρ is an l-th root of unit. Hence each fixed point p contributes

$$\sum_{k=1}^{l-1} \frac{1}{1-\rho^k} = \frac{1}{2}(l-1).$$

After summation we get

$$\frac{n}{2}(l-1) = l - 1 + l\Delta - (l-1)g(C),$$

which simplifies to the prescribed formula. □

3.4 For a singular curve C, we denote by $\nu : C^\nu \to C$ the normalization map and $\delta = \nu_* \mathcal{O}_{C^\nu}/\mathcal{O}_C$ the torsion sheaf supported on $\mathrm{Sing}(C)$. Denote $h^0(\delta) = p_a(C) - g(C^\nu)$, where $p_a(C)$ is the arithmetic genus of C.

Lemma 6. *An irreducible and reduced curve $C \equiv L$ on a fake projective plane M is a smooth curve of genus 3. For $C \equiv 2L$, $g(C^\nu) \geqslant 4$ and $h^0(\delta) \leq 2$.*

Proof. We first remark that for $C \subseteq M$, $g(C^\nu) \geq 2$ as M is hyperbolic. The usual Schwarz Lemma applied to the map induced by the nomalization $\nu : C^\nu \to M$ (cf. [CCL]) for the manifolds equipped with Poincaré metrics implies that the Kähler forms satisfy $\nu^*\omega_M \leqslant \omega_{C^\nu}$, with equality if and only if it is a holomorphic isometry leading to totally geodesic $\nu(C)$. Since there is no totally geodesic curve on a fake projective plane as mentioned in the proof of Lemma 4, the inequality is strict. Integrating over C^ν, we get

$$2k = \frac{2}{3}(K \cdot C) = \frac{2}{3}(\nu^* K \cdot C^\nu) = \nu^*\omega \cdot C^\nu < \omega_{C^\nu} \cdot C^\nu \leq 2p_a(C) - 2 = k(k+3),$$

where we used the fact that the Ricci curvature is $\frac{3}{2}$ of the holomorphic sectional curvature for the Poincaré metric on M. Hence $k = 1$ implies that $g(C^\nu) = 3$, $h^0(\delta) = 0$, and hence C is smooth. The second statement is proved similarly. □

Lemma 7. *Let M be a fake projective plane with $K_M \equiv 3L$. Assume that L is invariant under a non-trivial subgroup $H = C_3$ or C_7 of $\mathrm{Aut}(M)$. If Σ is an invariant section of $\mathrm{H}^0(M, 2L)$ as in Lemma 4, then Σ has an H-fixed point.*

Moreover, either one of the following happens:

(1) Σ integral, $g_a(\Sigma) = 6$, $g(\Sigma^\nu) \geq 4$, and $h^0(\delta) \leq 2$;
(2) $\Sigma = 2C$, and C is smooth of genus 3;
(3) $\Sigma = C_1 \cup C_2$, and C_i's are smooth of genus 3.

Proof. By Lemma 4, we consider three cases: $\Sigma = \Sigma_1 + \Sigma_2$, $\Sigma = 2C$, or Σ is irreducible and reduced.

If $\Sigma = \Sigma_1 + \Sigma_2$, then $\Sigma_1 \cap \Sigma_2 = \{p\}$ is a point. As any element of H carries an irreducible component of Σ to another irreducible component of Σ and $|H|$ is odd, Σ_i's are H-invariant and $p \in \Sigma$ is an H-fixed point.

If $\Sigma = 2C$, then $C \equiv L$ and is smooth of genus 3 by Lemma 6. If H acts without fixed point on C, then the quotient C/H is a Riemann surface of Euler-Poincáre number

$$\chi_{\text{top}}(C/H) = 2 - 2g(C/|H|) = \frac{-4}{|H|}.$$

This is certainly impossible for $|H| = 3$ or 7.

Suppose now that Σ is irreducible and reduced. The arithmetic genus $g_a(\Sigma)$ of Σ is given by

$$2(g_a(\Sigma) - 1) = 2\chi(\Sigma) = (K + 2L).(2L) = 10,$$

from which we conclude that $g_a(\Sigma) = 6$. Note that $g(\Sigma^\nu) \geq 4$ and $h^0(\delta) \leq 2$ from Lemma 6, where Σ^ν is the normalization of Σ.

If H acts without fixed point on Σ, then H acts without fixed point on Σ^ν. Hence Σ^ν/H is a Riemann surface of Euler-Poincáre number

$$\chi_{\text{top}}(\Sigma^\nu/H) = 2 - 2g(\Sigma^\nu/H) = \frac{-10 + 2h^0(\delta)}{|H|}.$$

and

$$0 \leq g(\Sigma^\nu/|H|) \leq 1 + 5/|H|.$$

If $|H| = 7$, then $g(\Sigma^\nu/H) \leq 1$. It follows that an entire holomorphic map from $\mathbb{C}$ to Σ^ν/H lifts to the unramified covering Σ^ν which maps into M, contradicting the fact that M is hyperbolic.

If $|H| = 3$, then there is only one possibility that $g(\Sigma^\nu/H) = 2$ and $h^0(\delta) = 2$. In particular, there is at least one singular point $P \in \Sigma$. If now H acts on Σ freely, then there are at least three singular points as the H-orbit of P and $h^0(\delta) \geq 3$. This is absurd. □

4. The case of $\mathrm{Aut}(M) = C_7 : C_3$

4.1. The goal of this section is to apply our argument to the case of a fake projective plane M with $\mathrm{Aut}(M) = C_7 : C_3$, which gives an alternate approach to such cases dealt with in [GKMS].

4.2 We prove the first part of our Main Theorem which is done in [GKMS] by different method. We start with a lemma about plane curve singularities, which should be well-known but we can not find a good reference.

Lemma 8. *Let (Σ, o) be a germ of reduced curve with b irreducible branches $\Sigma_1, \ldots, \Sigma_b$. Denote by Σ_i^ν the normalization of each irreducible branches.*

(a) *For* $\delta = \nu_* \mathcal{O}_{\Sigma^\nu}/\mathcal{O}_\Sigma$ *and* $\delta_i = \nu_* \mathcal{O}_{\Sigma_i^\nu}/\mathcal{O}_{\Sigma_i}$ *supported over the singularity* $o \in \Sigma$,

$$h^0(\delta) \geq (b-1) + \sum_{i=1}^{b} h^0(\delta_i).$$

(b) $h^0(\delta) = b-1$ *if and only if all the branches are smooth and*

$$\mathcal{O}_{\Sigma,o} \cong \frac{k[[t_1, \ldots, t_b]]}{(\{t_i t_j : \ i \neq j\})}.$$

(c) *If* $h^0(\delta) = 1$, *then either* $b = 1$ *and* $o \in \Sigma$ *is cuspidal or* $b = 2$ *and* $o \in \Sigma$ *is a normal crossing (analytically a node).*

Proof. The statements (a) follows straight forward from the exact sequences

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{O}_{\Sigma,o} & \to & \oplus \mathcal{O}_{\Sigma_i,o} & \to & \tilde{\delta} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}_{\Sigma,o} & \to & \oplus \mathcal{O}_{\Sigma_i^\nu,o} \cong \oplus k[[t_i]] & \to & \delta & \to & 0 \\
 & & & & \downarrow & & & & \\
 & & & & \oplus \delta_i & & & &
\end{array}$$

and the snake lemma: $b - 1 \leq h^0(\tilde{\delta}) = h^0(\delta) - \sum_i h^0(\delta_i)$. Here $h^0(\tilde{\delta}) \geq b-1$ as it contains the k-vector space generated by $e_i/(1, \ldots, 1)$, where e_i is the unit of $\mathcal{O}_{\Sigma_i,o}$ and $(1, \ldots, 1)$ is the image of the unit of $\mathcal{O}_{\Sigma,o}$.

For (b), $h^0(\delta) = b - 1$ if and only if $h^0(\delta_i)$'s are zero. It is clear from the above discussion that this happens exactly when $m_{\Sigma,o}$ maps surjectively onto (t_i) for each i. This is the same as saying that Σ is a normal crossing.

The statement (c) is a local computation. From the assumption, we know from (a) that $b \leq 2$. If $b = 1$, then there is a sequence

$$0 \to \mathcal{O}_{\Sigma,o} \to k[[t]] \to \delta \to 0$$

with δ a zero dimensional sheaf of length one. As an k-algebra, it is only possible that $\mathcal{O}_{\Sigma,o} \cong k[[t^2, t^3]]$ and we get

$$\mathcal{O}_{\Sigma,o} \cong \frac{k[[x,y]]}{(x^2 - y^3)}.$$

If $b = 2$, then then there is a sequence

$$0 \to \mathcal{O}_{\Sigma,o} \to k[[t_1]] \oplus k[[t_2]] \to \delta \to 0.$$

If $h^0(\delta) = 1$, then it is only possible that after suitable change of coordinates

$$k[[t_1]] \oplus k[[t_2]] \supseteq \mathcal{O}_{\Sigma,o} = k[[t_1, 0), (0, t_2)]] \cong \frac{k[[x,y]]}{(xy)}.$$

□

Remark 1. *It is possible that* $h^0(\tilde{\delta}) > b - 1$. *This happens if and only if* $h^0(\delta) > \sum_i h^0(\delta_i)$. *The difference comes from the gluing of* Σ_i*'s along* $o \in \Sigma$.

Theorem 1. *Let* M *be a fake projective plane with* $\mathrm{Aut}(M) = C_7 : C_3$ *and* $K_M = 3L$. *Then the sequence* $\mathcal{O}_M, -L, -2L$ *forms an exceptional collection.*

Proof. From the definition of exceptional collection, we have to verify $h^i(M,L) = 0$, $i = 0,1,2$. It is enough to show that $h^2(M,L) = h^0(M,2L) = 0$. Indeed, $h^0(M,L) = 0$ follows immediately and $h^1(M,L) = 0$ by $\chi(M,L) = 0$. Suppose now that $h^0(M,2L) \neq 0$.

Consider $H = C_7 < \mathrm{Aut}(X)$, the unique 7-Sylow subgroup, and by Lemma 2 the space of sections $\mathrm{H}^0(M,2L)$ is H-invariant. There is an H-invariant section $\Sigma \in \mathrm{H}^0(M,2L)$ by Lemma 4 and an H-fixed point by Lemma 6. Moreover, Σ is either irreducible and reduced, or $\Sigma = 2C$, or $\Sigma = \Sigma_1 + \Sigma_2$ is reducible.

Observe that $\mathrm{Fix}(\Sigma) = \mathrm{Fix}(M) \cap \Sigma$. In particular, $|\mathrm{Fix}(\Sigma)| \leq 3$ by the work of [K].

For the induced action of H on Σ^ν, we denote by $x = \dim_{\mathbb{C}} \mathrm{H}^1(\mathcal{O}_{\Sigma^\nu})^{\mathrm{inv}}$ the dimension of H-invariant 1-forms and $n = |\mathrm{Fix}(\Sigma^\nu)|$ the number of H-fixed points on Σ^ν.

Case 1: Σ is irreducible and reduced. Here $g_a(\Sigma) = 6 = g(\Sigma^\nu) + h^0(\delta)$.

Assume first that $\Sigma = \Sigma^\nu$, then $g(\Sigma) = 6$ and $n \leq 3$. For $l = 7$, Lemma 5 implies that $3n + 7x = 12$ and $(n,x) = (4,0)$. This contradicts to the inequality $n \leq 3$.

Assume now that $\Sigma \neq \Sigma^\nu$. Applying Lemma 5 to the lifted action of H on Σ^ν, the normalization of Σ, with $l = |H| = 7$, we get

$$3n + 7x + h^0(\delta) = 12.$$

Since fake projective planes are hyperbolic, $g(\Sigma^\nu) \geq 2$ and hence $1 \leq h^0(\delta) \leq 4$. We study case by case.

If $h^0(\delta) = 1$, then $3n + 7x = 11$ and there is no nonnegative integer solution.

If $h^0(\delta) = 2$, then $3n + 7x = 10$ and $(n,x) = (1,1)$. From the holomorphic Lefschetz fixed point theorem, we have

$$\frac{1}{1-\eta} + \xi_1 + \xi_2 + \xi_3 = 0,$$

where $\eta, \xi_j \in (\mathbb{Z}/7\mathbb{Z})^\times$. It can be checked directly from Matlab that there is no solution to the above equation.

If $h^0(\delta) = 3$, $3n + 7x = 9$ and $(n,x) = (3,0)$. From the holomorphic Lefschetz fixed point theorem, we have

$$\frac{1}{1-\eta_1} + \frac{1}{1-\eta_2} + \frac{1}{1-\eta_3} + \xi_1 + \xi_2 + \xi_3 = 1,$$

where $\eta_i, \xi_j \in (\mathbb{Z}/7\mathbb{Z})^\times$. It can be checked directly from Matlab that there is no solution to the above equation.

If $h^0(\delta) = 4$, then $3n + 7x = 8$ and there is no nonnegative integral solution.

Case 2: $\Sigma = \Sigma_1 + \Sigma_2$ is reducible with two irreducible components Σ_i's. Denote $\Sigma^\nu = C_1 \sqcup C_2$ the normalization of Σ. Here C_i's are irreducible components of Σ^ν and are the normalization of Σ_i's respectively.

For $g \in H = C_7$, if $gC_1 = C_2$, then $gC_2 = C_1$. As $|H|$ is odd, $C_1 = g^{|H|}C_1 = C_2$. This is impossible and hence H acts on C_i's. For $g_i = g(C_i)$,

$$6 = g_a(\Sigma) = g_1 + g_2 + h^0(\delta) - 1,$$

where again for $\nu : \Sigma^\nu \to \Sigma$ the normalization map, $\delta = \nu_* \mathcal{O}_{\Sigma^\nu}/\mathcal{O}_\Sigma$ is a torsion sheaf supporting on $\mathrm{Sing}(\Sigma) \neq \emptyset$. Since M is hyperbolic, $g(C_i) \geq 2$ and hence $1 \leq h^0(\delta) \leq 3$.

Denote n_i the number of H-fixed points on C_i. For $l = |H|$, by Lemma 5

$$n_i(l-1) + 2lx_i = 2(l-1) + 2g_i,$$

where $x_i = \dim_{\mathbb{C}} \mathrm{H}^1(\mathcal{O}_{C_i})^{\mathrm{inv}}$. Hence for $n = n_1 + n_2$ and $x = x_1 + x_2$,

$$n(l-1) + 2lx + 2h^0(\delta) = 4(l-1) + 14.$$

For $l = |H| = 7$, we get $3n + 7x + h^0(\delta) = 19$. If a singular point of Σ is not fixed, then $h^0(\delta) \geq 7$. But we have seen that $1 \leq h^0(\delta) \leq 3$. Hence all singularities of Σ are fixed. Denote by $p = \Sigma_1 \cap \Sigma_2$ the unique intersection point and write $\delta = \delta_p + \delta'$ for obvious reason, we have $h^0(\delta_p) \geq 1$.

If $h^0(\delta) = 1$, then $h^0(\delta_p) = 1$ and p is the unique singularity of $\Sigma = \Sigma_1 \cup \Sigma_2$. By Lemma 7, $p \in \Sigma$ is nodal and lifts to two fixed points on Σ^ν. Hence $n \geq 2$, $3n + 7x = 18$, and $(n, x) = (6, 0)$ is the only solution. But then apart from the two fixed points above p, there should be four more fixed points on Σ. This is a contradiction to $|\mathrm{Fix}(\Sigma)| \leq 3$.

If $h^0(\delta) = 2$, then either $h^0(\delta_p) = h^0(\delta') = 1$ or $h^0(\delta_p) = 2$. In the former case, either Σ has exactly two nodal singularities which lift to four fixed points on Σ^ν, or Σ has one node at p and a cusp at another point. Hence we have $n \geq 3$ for $3n + 7x = 17$ and there is no nonnegative integral solution.

In the later case where $h^0(\delta_p) = 2$, if $b = 3$ over $p \in \Sigma$, then $p \in \Sigma$ has embedded dimension 3 by Lemma 7 (b). This contradicts to the fact that $\Sigma \subseteq X$. Hence $b = 2$ and there are two fixed points over p. But now $n \geq 2$ and there is no nonnegative integral solution to $3n + 7x = 17$.

If $h^0(\delta) = 3$, then $3n + 7x = 16$ and $(n, x) = (3, 1)$. Since $x_1 + x_2 = 1$, $n_i \geq 1$, and $g_i \geq 2$, it is easy to see that there is no nonnegative integral solution to the system of linear equations

$$\begin{cases} 3n_1 - g_1 + 7x_1 = 6, \\ 3n_2 - g_2 + 7x_2 = 6, \\ n_1 + n_2 = 3. \end{cases}$$

Case 3: $\Sigma = 2C$ with C an irreducible and reduced curve.

Here $C \equiv L$ and $3 = g_a(C) = g(C^\nu) + h^0(\delta)$. Moreover, $g(C^\nu) \geq 2$ as M is hyperbolic. We consider two cases: $(g(C^\nu), h^0(\delta)) = (3, 0)$ or $(2, 1)$.

Suppose that $(g(C^\nu), h^0(\delta)) = (2, 1)$. For $l = 7$, $3n + 7x = 8$ by Lemma 5 applied to C. There is no integer solution.

Suppose that $(g(C^\nu), h^0(\delta)) = (3, 0)$ and hence $C = C^\nu$. Since $l = 7$, $3n + 7x = 9$ by Lemma 5 applied to C. It is only possible that $(n, x) = (3, 0)$ and there are three smooth fixed points on C. From the holomorphic Lefschetz fixed point theorem, we have

$$\frac{1}{1-\eta_1} + \frac{1}{1-\eta_2} + \frac{1}{1-\eta_3} + \xi_1 + \xi_2 + \xi_3 = 1,$$

where $\eta_i, \xi_j \in (\mathbb{Z}/7\mathbb{Z})^\times$. It can be checked directly from Matlab that there is no solution to the above equation.

We conclude that it is only possible $\mathrm{H}^0(M, 2L) = 0$ and hence the existence of the required exceptional collection. $\square$

5. The case of $\mathrm{Aut}(M) = C_3 \times C_3$

5.1 In this section, we prove the second part of the Main Theorem. From now we assume that M is a fake projective plane with $\mathrm{Aut}(M) = C_3 \times C_3$. We refer the readers to the list in the Main Theorem.

We have shown in Lemma 1 and Lemma 2 that there is an ample line bundle L with $K_M = 3L$, where L is $\mathrm{Aut}(M)$-invariant and $\mathrm{H}^0(M, 2L)$ is an $\mathrm{Aut}(M)$-module. We aim to show that $h^0(M, 2L) \neq 0$ is impossible as in Section **4** by investigating the geometry of an invariant curve $C \equiv 2L$. We first study the local geometry of singularities.

Lemma 9. *Let (C, o) be an analytical germ of a reduced singular plane curve. Let $\nu : C^\nu \to C$ be the normalisation and $\delta = \mathcal{O}_C/\nu_*\mathcal{O}_{C^\nu}$ be the zero dimensional sheaf supported at the unique singularity $o \in C$. Let $h^0(\delta)$ be the length of δ and $X_1, \ldots, X_r$ be the irreducible branches of C at $o \in C$. Then*

$$h^0(\delta) = \sum_i h^0(\delta_i) + \sum_{i<j} (X_i.X_j).$$

In particular, $h^0(\delta) \geq r(r-1)/2$.

Furthermore, suppose that $H = C_3$ acts on $\mathbb{C}^2$ of weight $\frac{1}{3}(1,2)$ with (C, o) an invariant curve such that $o \in C$ descends to the unique singularity $\hat{o} \in \mathbb{C}^2/G$. If the induced action on (C, o) is nontrivial and $h^0(\delta) \leq 2$, then either

- *(a) $h^0(\delta) = 1$ and $o \in C$ is a node , or*
- *(b) $h^0(\delta) = 2$ and $o \in C$ is equivalent to* $\mathrm{Spec}(\frac{k[[x,y]]}{(x(x-y^2))})$.

In both cases, $r = 2$ and $o \in C$ lifts to two H-fixed points on the normalisation C^ν.

Proof. Part (a) is given by Hironaka [Hi]. For part (b), we first observe that $h^0(\delta) \leq 2$ implies that $r \leq 2$.

Suppose that $r = 1$. We consider the sequence

$$0 \to \mathcal{O}_{C,o} \cong \frac{k[[x,y]]}{(f(x,y))} \xrightarrow{\phi} k[[t]] \to \delta \to 0,$$

where $\phi(x) = u(t) = \sum_{m \geq 0} u_m t^m$ and $\phi(y) = v(t) = \sum_{n \geq 0} v_n t^n$. Here we choose (x, y) to be H-invariant coordinates with $\omega \cdot x = \omega x$ and $\omega \cdot y = \omega^2 y$, where $\omega = \exp(2\pi i/3)$. Assume that $\omega \cdot t = \omega^\alpha t$ for $\alpha \in \{1, 2\}$. Since ϕ is H-invariant, we have

$$\begin{cases} \omega u(t) = \phi(\omega x) = \phi(\omega \cdot x) = \omega \cdot u(t) = \sum_{m \geq 0} u_m \omega^{\alpha m} t^m \\ \omega^2 v(t) = \phi(\omega^2 y) = \phi(\omega \cdot y) = \omega \cdot v(t) = \sum_{n \geq 0} v_n \omega^{\alpha n} t^n \end{cases}.$$

Since this happens for infinitely many $m, n \in \mathbb{N}$, we have

$$\alpha m \equiv 1, \ \alpha n \equiv 2 \mod 3.$$

Assume that $\alpha = 1$, then we can write

$$u(t) = t(\sum_{k\geq 0} a_k t^{3k}),\ v(t) = t^2(\sum_{l\geq 0} b_l t^{3l}),$$

where $a_k = u_{3k+1}$ and $b_l = v_{3l+2}$.

Suppose that $a_0 \neq 0$, then $u'(0) \neq 0$ and hence C is smooth while we assume C is singular to start with. So $a_0 = 0$ and we write

$$u(t) = t^4(\sum_{k\geq 0} a'_k t^{3k}).$$

If $b_0 \neq 0$, then $v(t) = t^2 \cdot \text{unit}$. Hence for $a'_0 = 0$ or a change of coordinates for $a'_0 \neq 0$, the k-algebra $\mathcal{O}_{C,o}$ has to be $k[[t^2 \cdot \text{unit}, t^{4+3k} \cdot \text{unit}]]$ with $k \geq 1$. But then δ contains at least t, t^3, t^5, which contradicts to $h^0(\delta) \leq 2$. The case $\alpha = 2$ is similar. Hence we must have $r > 1$.

Suppose now that $r = 2$ and $h^0(\delta) = h^0(\delta_1) + h^0(\delta_2) + X_1.X_2 \geq X_1.X_2 \geq 1$. If $h^0(\delta) = 1$, then X_i's are smooth and intersect transversally. This is the nodal case (a). If $h^0(\delta) = 2$, then say $h^0(\delta_i) = 0$ for $i = 1, 2$ and $X_1.X_2 = 2$. In this case, we can assume that $X_1 = \{x = 0\}$ after a change of coordinates. Hence X_2 can be normalised to $x - y^2 = 0$ and we get $C = \{x(x - y^2) = 0\}$, which is case (b). On the other hand, if $h^0(\delta_1) + h^0(\delta_2) = 1$ and we assume that X_1 is smooth, then $X_1.X_2 \geq 2$ as X_2 with $h^0(\delta_2) = 1$ is singular. This is absurd.

□

Lemma 10. *Let M be a fake projective plane with $K_M \equiv 3L$ where L is invariant under a nontrivial subgroup $H = C_3$ of the automorphism group of M. Let $C \equiv kL$ be an integral H-invariant curve with $k = 1$ or 2. For $\nu : C^\nu \to C$ the normalisation map, denote by n the number of H-fixed points on Σ^ν, $h^0(\delta)$ the length of $\nu_*\mathcal{O}_{C^\nu}/\mathcal{O}_C$, and $x = dim_{\mathbb{C}}\mathrm{H}^1(C, \mathcal{O})^{\mathrm{inv}}$. Then there is a finite list of C according to $(n, h^0(\delta), x)$:*

- (N) $(n, h^0(\delta), x) = (2, 0, 1)$*: $C \equiv L$ is smooth and has two fixed smooth fixed points;*
- (I_1) $(n, h^0(\delta), x) = (2, 0, 2)$*: $C \equiv 2L$ is a smooth curve of $g(C) = 6$ and has two smooth fixed points;*
- (I_2) $(n, h^0(\delta), x) = (4, 1, 1)$*: $C \equiv 2L$ has one fixed node, which is the unique singularity of C, and two fixed smooth points;*
- (I_3) $(n, h^0(\delta), x) = (3, 2, 1)$*: $C \equiv 2L$ has one fixed singularity of type* $\mathrm{Spec}(\frac{k[[x,y]]}{(x(x-y^2))})$, *which is the unique singularity of C, and one fixed smooth point.*

Proof. Let's summarise a table with known information from Lemma 4, 5, 6, 7:

$C \equiv L_k$	$p_a(C)$	$g(C^\nu) = p_a(C) - h^0(\delta)$	Identity(1)
$k = 1$	3	3	$n + 3x = 5$
$k = 2$	6	$6 - h^0(\delta)$	$n + h^0(\delta) + 3x = 8$

Here $n \geq 1$ and $h^0(\delta) \leq 2$. Moreover, since H acts on $\mathrm{Sing}(C)$ and $h^0(\delta) \leq 2$, $|\mathrm{Sing}(C)| \leq 2$ and all the singularities are H-fixed points of C. Also we remark that from [CS] and [K], the number $|Fix(C)|$ of fixed points on C, which from the construction is the same as the number of fixed points of M on C, is at most $|\mathrm{Fix}(M)| = 3$.

If $C \equiv L$, then $n + 3x = 5$. Since $C = C^\nu \subseteq M$ and $n \leq |\mathrm{Fix}(M)| = 3$, there is only one solution $(n, x) = (2, 1)$. Note that there is no contradiction to holomorphic Lefschetz fixed point theorem as

$$\frac{1}{1-\omega} + \frac{1}{1-\omega^2} + \omega + \omega^2 = 0,$$

where $\omega = \exp(\frac{2\pi i}{3})$. This is the case (N).

If $C \equiv 2L$, then $n + h^0(\delta) + 3x = 8$ and we have the following possible solutions:

singularity	Identity(1)	(n, x)
$h^0(\delta) = 0$	$n + 3x = 8$	$(2, 2)$ as $n \leq \|\mathrm{Fix}(M)\| = 3$
$h^0(\delta) = 1$	$n + 3x = 7$	$(7, 0), (4, 1), (1, 2)$
$h^0(\delta) = 2$	$n + 3x = 6$	$(6, 0), (3, 1)$

A smooth curve $C \equiv 2L$ with $(n, x) = (2, 2)$ is the case (I_1).

By Lemma 9, $h^0(\delta) = 1$ only occurs when it is a node and it lifts to two H-fixed points on C^v. Hence $n \geq 2$. For $(n, x) = (7, 0)$, outside $\mathrm{Sing}(C)$ there must be 5 smooth fixed points on C, contradicting to $|\mathrm{Fix}(C)| \leq 3$. Hence $(4, 1)$ is the only solution and this is case (I_2).

If now $h^0(\delta) = 2$, then $|\mathrm{Sing}(C)| = 1$, or 2. If there are two singular points, then these are two C_3-fixed nodes which by Lemma 9 lifts in total to four C_3-fixed points on C^ν and $n \geq 4$. Hence $(6, 0)$ is the only solution. But then there must be two more smooth C_3-fixed points on C and this contradicts to $|\mathrm{Fix}(C)| \leq 3$. If there is only one singular point, then by Lemma 9 it is locally $\mathrm{Spec}(\frac{k[[x,y]]}{(x(x-y^2))})$, which lifts to two C_3-fixed points on C^ν and $n \geq 2$. But then $(6, 0)$ is impossible as there must be four more smooth C_3-fixed points, which contradicts to $|\mathrm{Fix}(C)| \leq 3$. Hence $(n, x) = (3, 1)$ and this is case (I_3). Note that the holomorphic Lefschetz fixed point theorem has a solution,

$$\frac{1}{1-\omega} + \frac{1}{1-\omega} + \frac{1}{1-\omega} + \omega + \omega^2 + \omega^2 = 0.$$

□

Theorem 2. *Let M be a fake projective plane with $Aut(M) = C_3 \times C_3$. Let L be the $Aut(M)$-invariant line bundle with $K_M = 3L$ from Lemma 2. Then $\mathcal{O}_M, -L, -2L$ forms an exceptional collection.*

Proof. As in Theorem 1, it is enough to show that $H^0(M, 2L) = 0$.

Suppose that $H^0(M, 2L) \neq 0$ and let Σ be an $\mathrm{Aut}(M)$-invariant section. We recall that from [CS] and [K], there are 4 singularities of type $\frac{1}{3}(1, 2)$ on $M/\mathrm{Aut}(M)$. In fact, we note that $\mathrm{Aut}(M)$ has four subgroups of C_3, denoted by $G_1, \ldots, G_4$. There are twelve points $P_i, i = 1, \ldots, 12$ of M with stabilizer G_j for some $j = 1, \ldots, 4$. The image of P_i in the quotient $N = M/\mathrm{Aut}(M)$ is a singularity of type $\frac{1}{3}(1, 2)$.

Let G_1 be the first C_3 factor and G_2 be the second C_3 factor.

Consider G_1-action on Σ. From Lemma 7 and Lemma 10, there are three possibilities:

(1) Σ is integral and has at most two smooth fixed points.
(2) $\Sigma = 2C$ and C is smooth of genus 3 with two smooth G_1-fixed points.
(3) Σ is reduced with two smooth components C_1 and C_2 of genus 3. Moreover, G_1 acts on each component C_i with two smooth G_1-fixed points.

Since Σ is $\mathrm{Aut}(M)$-invariant, the curve Σ in (1) and C in (2) are invariant under G_2. In case (3), as $\Sigma = C_1 \cup C_2$ has only two components, both C_i's are also invariant under G_2.

Since each curve, Σ, C, or C_i, has at least one smooth fixed G_1 point P and G_2-action permutes G_1-fixed points, as the G_2-orbit of P, there should be at least three G_1-fixed smooth points on each curve. This is a contradiction. □

5.4 The Main Theorem is the combination of Theorem 1 and Theorem 2. We finish with the proof of the Main Theorem'.

Proof.(of Main Theorem') For $\mathcal{O}_M, -L_1, -2L_2$ in the Main Theorem' to form an exceptional collection, we need to show that

$$h^i(M, L_1) = h^i(M, 2L_2) = h^i(M, 2L_2 - L_1) = 0, \ i = 0, 1, 2.$$

We consider vanishing of $h^i(M, L_1)$ first. Note that $h^2(M, L_1) = h^0(M, K_M - L_1)$. Since both L_1 and $K_M - L_1$ are invariant under $\mathrm{Aut}(M)$,

$$h^0(M, L_1) = h^0(M, K_M - L_1) = 0$$

by the same proof as in the Main Theorem. It then follows that $h^1(M, L_1) = 0$ by the Riemann-Roch formula. The other vanishing are proved similarly. □

Acknowledgment This work is partially done during the first author's visit at National Center of Theoretical Science (NCTS/TPE) and National Taiwan University at Taiwan, and the second authors visit of the Institute of Mathematics of the University of Hong Kong. The authors thank the warm hospitality of the institutes.

References

[AO] Alexeev, V., Orlov, D., Derived categories of Burniat surfaces and exceptional collections. (English summary) Math. Ann. 357 (2013), no. 2, 743-759.

[BvBS] Böhning, C.; von Bothmer, H-C; Sosna, P., On the derived category of the classical Godeaux surface, Adv. Math. 243 (2013), 203-231.

[BHPV] Barth, W. P., Hulek, K., Peters, C. A. M., Van de Ven, A., Compact complex surfaces. Second edition. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics 4. Springer-Verlag, Berlin, 2004.

[CS] Cartwright, D., Steger, T., Enumeration of the 50 fake projective planes, C. R. Acad. Sci. Paris, Ser. 1, 348 (2010), 11-13, see also
http://www.maths.usyd.edu.au/u/donaldc/fakeprojectiveplanes/.

[CCL] Chen, Z., S. Y. Cheng, S. Y., and Lu, Q. K., On the Schwarz lemma for complete Kähler manifolds, Sci. Sinica 22 (1979), no. 11, 1238-1247.

[F] Fakhruddin, N., Exceptional collections on 2-adically uniformised fake projective planes, arXiv:1310.3020.

[GS] Galkin, S., Shinder, E., Exceptional collection on the Beauville surface, preprint (2012), arXiv:1210.3339.

[Hi] Hironka, H., On the Arithmetic Genera and the Effective Genera of Algebraic Curves, Mere. Coll. Sci., Univ. of Kyoto 30, 177-195 (1957)

[K] Keum, J., Quotients of fake projective planes, Geom Topol, 2008, 2497-2515. (arXiv:0802.3435)

[GKMS] Galkin, S., Katzarkov, L., Mellit A., Shinder, E., Minifolds and phantoms, preprint arXiv:1305.4549, (2013).

[GH] Griffiths, P., Harris, J., Principle of Algebraic Geometry, New York, Wiley 1978.

[GO] Gorchinskiy, S, Orlov, D., Geometric phantom categories. (English summary), Publ. Math. I. H. E. S. 117 (2013), 329-349.

[P] Peters, C.A.M., Holomorphic automorphisms of compact Kähler surfaces and their induced actions in cohomology, Invent. Math. 52 (1979), 143-148.

[PY] Prasad, G., and Yeung, S.-K., Fake projective planes. Inv. Math. 168(2007), 321-370; Addendum, ibid 182(2010), 213-227.

[RV] Remmert, R., Van de Ven, A., Zur Funktionentheorie homogener komplexer Mannigfaltigkeiten, Topology 2 (1963), 137-157.

[Y1] Yeung, S.-K., Integrality and arithmeticity of co-compact lattices corresponding to certain complex two-ball quotients of Picard number one, Asian J. of Math., 8(2004), 104-130; Erratum, ibid, 13(2009), 283-286.

[Y2] Yeung, S.-K., Exotic structures arising from fake projective planes, Sci. China Math. 56 (2013), 43-54.

[Y3] Yeung, S.-K., Classification of surfaces of general type with Euler number 3, J. Reine Angew. Math. 679 (2013), 1-22,
(see also http://www.math.purdue.edu/%7Eyeung/papers/fake-euler.pdf)

Mathematics Department, Purdue University, West Lafayette, IN 47907 USA
E-mail address: `lai37@math.purdue.edu`

Mathematics Department, Purdue University, West Lafayette, IN 47907 USA
E-mail address: `yeung@math.purdue.edu`