Interpretations of Solving Linear Systems

2×2: Solve: $\begin{cases} 2x - y = 4 & \text{——— ①} \\ x + 3y = 9 & \text{——— ②} \end{cases}$

Solve by elimination & substitution:

① $\Longrightarrow$ $y = 2x - 4$

② $\Longrightarrow$ $x + 3(2x-4) = 9$

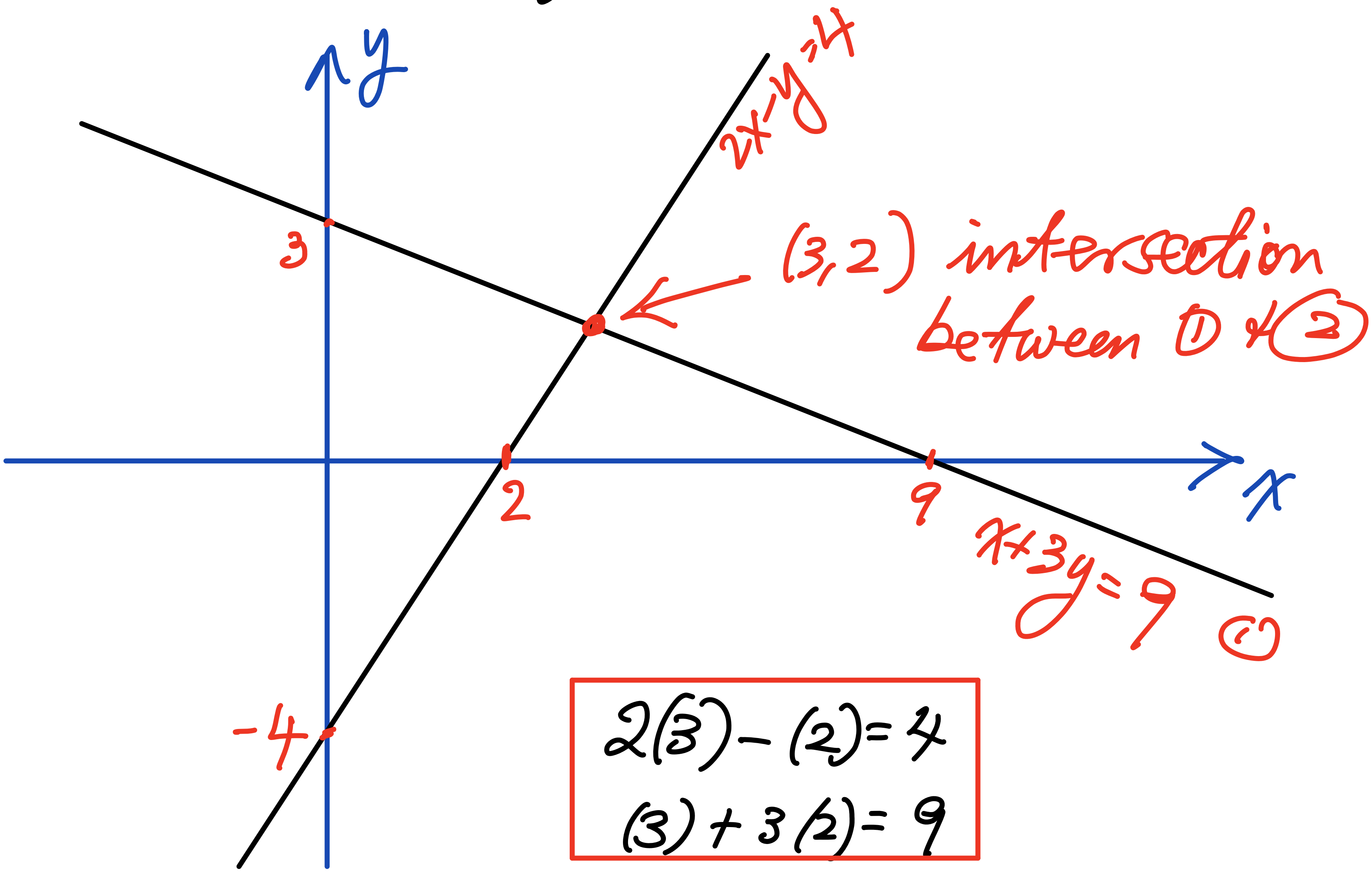
$7x = 9 + 12 = 21$

Hence $x = 3$, $y = 2(3) - 4 = 2$

Solution: $(x, y) = (3, 2)$, or $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

(I) Finding intersection between 2 st. lines

(Look at each equation or row)



$$2(3)-(2)=4$$
$$(3)+3(2)=9$$

(II) Writing vector as Linear Combination (Look at "columns")

$$\begin{pmatrix} 2x - y = 4 \\ x + 3y = 9 \end{pmatrix}$$

vector addition

$$\begin{pmatrix} 2x \\ x \end{pmatrix} + \begin{pmatrix} -y \\ 3y \end{pmatrix} = \begin{pmatrix} 4 \\ 9 \end{pmatrix}$$

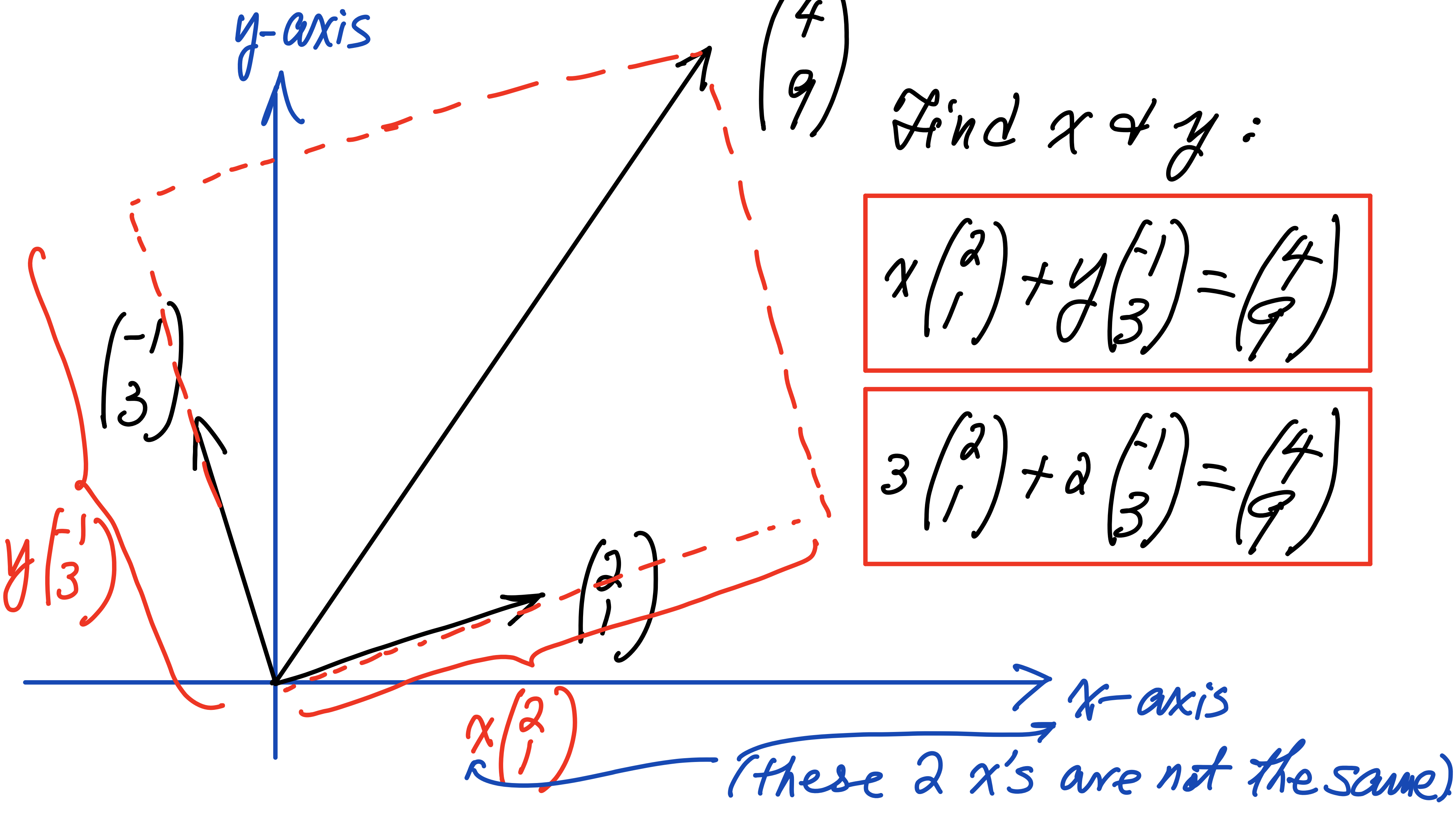
scalar multiplication

$$x\begin{pmatrix} 2 \\ 1 \end{pmatrix} + y\begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 9 \end{pmatrix}$$

unknowns

(II) Writing vector as Linear Combination

(Look at "columns")



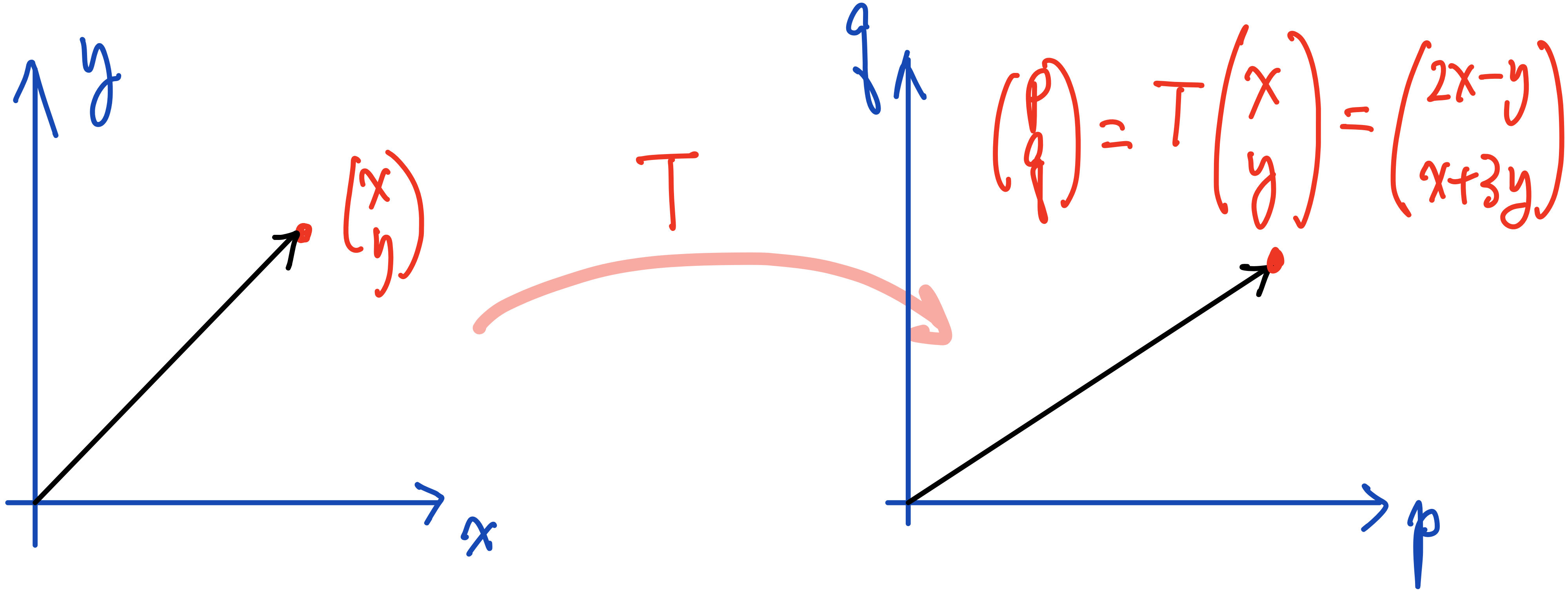
Find x & y:

$$\boxed{x\begin{pmatrix}2\\1\end{pmatrix}+y\begin{pmatrix}-1\\3\end{pmatrix}=\begin{pmatrix}4\\9\end{pmatrix}}$$

$$\boxed{3\begin{pmatrix}2\\1\end{pmatrix}+2\begin{pmatrix}-1\\3\end{pmatrix}=\begin{pmatrix}4\\9\end{pmatrix}}$$

(III) Linear Transformation of Vectors

$$\begin{pmatrix} x \\ y \end{pmatrix} \xrightarrow{T} T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x-y \\ x+3y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$$



(III) Linear Transformation of Vectors

$$\begin{pmatrix} x \\ y \end{pmatrix} \xrightarrow{T} T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x-y \\ x+3y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$$

Solving $\begin{cases} 2x-y = 4 \\ x+3y = 9 \end{cases}$ is equivalent

to finding $\begin{pmatrix} x \\ y \end{pmatrix}$ such that $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 9 \end{pmatrix}$

(ie. $\begin{pmatrix} 4 \\ 9 \end{pmatrix}$ is the image of $\begin{pmatrix} x \\ y \end{pmatrix}$ under T.)

(III) Linear Transformation of Vectors

$$\begin{pmatrix} x \\ y \end{pmatrix} \xrightarrow{T} T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x-y \\ x+3y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$$

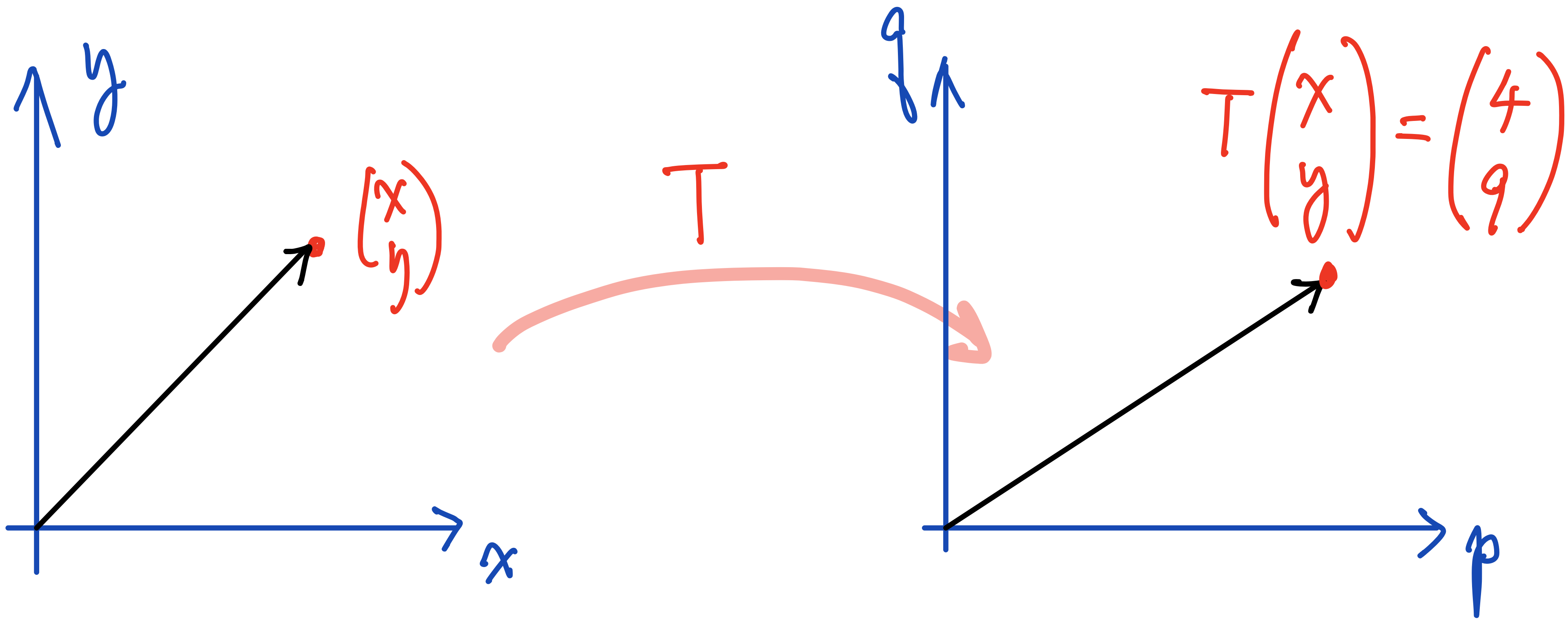
Solving $\begin{cases} 2x-y = 4 \\ x+3y = 9 \end{cases}$ is equivalent

to finding $\begin{pmatrix} x \\ y \end{pmatrix}$ such that $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 9 \end{pmatrix}$

(ie. $\begin{pmatrix} x \\ y \end{pmatrix}$ is the pre-image of $\begin{pmatrix} 4 \\ 9 \end{pmatrix}$ under T.)

(III) Linear Transformation of Vectors

$$\begin{pmatrix} x \\ y \end{pmatrix} \xrightarrow{\quad T \quad} T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x - y \\ x + 3y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$$

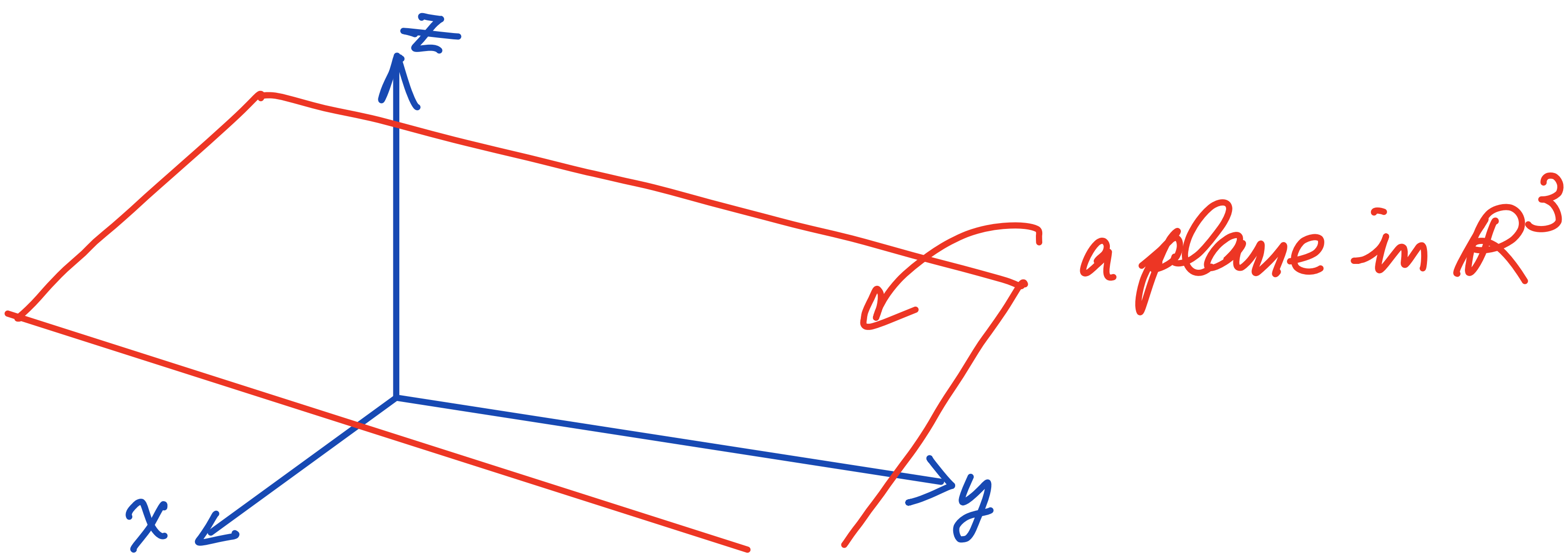


Example in $\mathbb{R}^3$

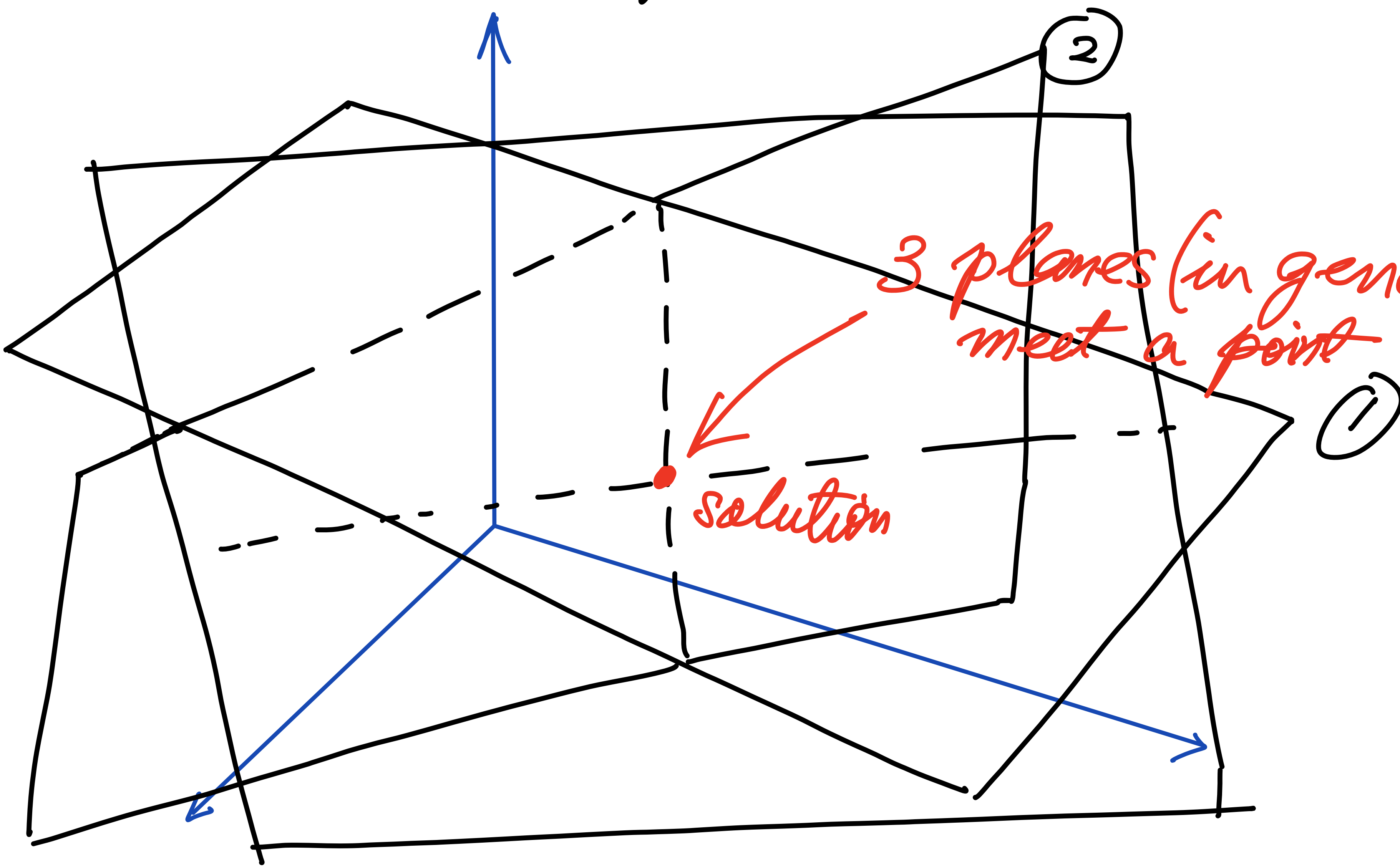
3×3 system:

$$(*)\quad \begin{cases} x+2y+z=1 & \text{— ①} \\ 3x+y+4z=0 & \text{— ②} \\ 2x+2y+3z=2 & \text{— ③} \end{cases}$$

$\boxed{\text{Each equation represents a plane in } \mathbb{R}^3}$



(I) Solving (*) is to find the intersection between 3 planes in $\mathbb{R}^3$.



(II) Writing Vector as linear combination

$$\begin{pmatrix} x+2y+z=1 \\ 3x+y+4z=0 \\ 2x+2y+3z=2 \end{pmatrix}$$

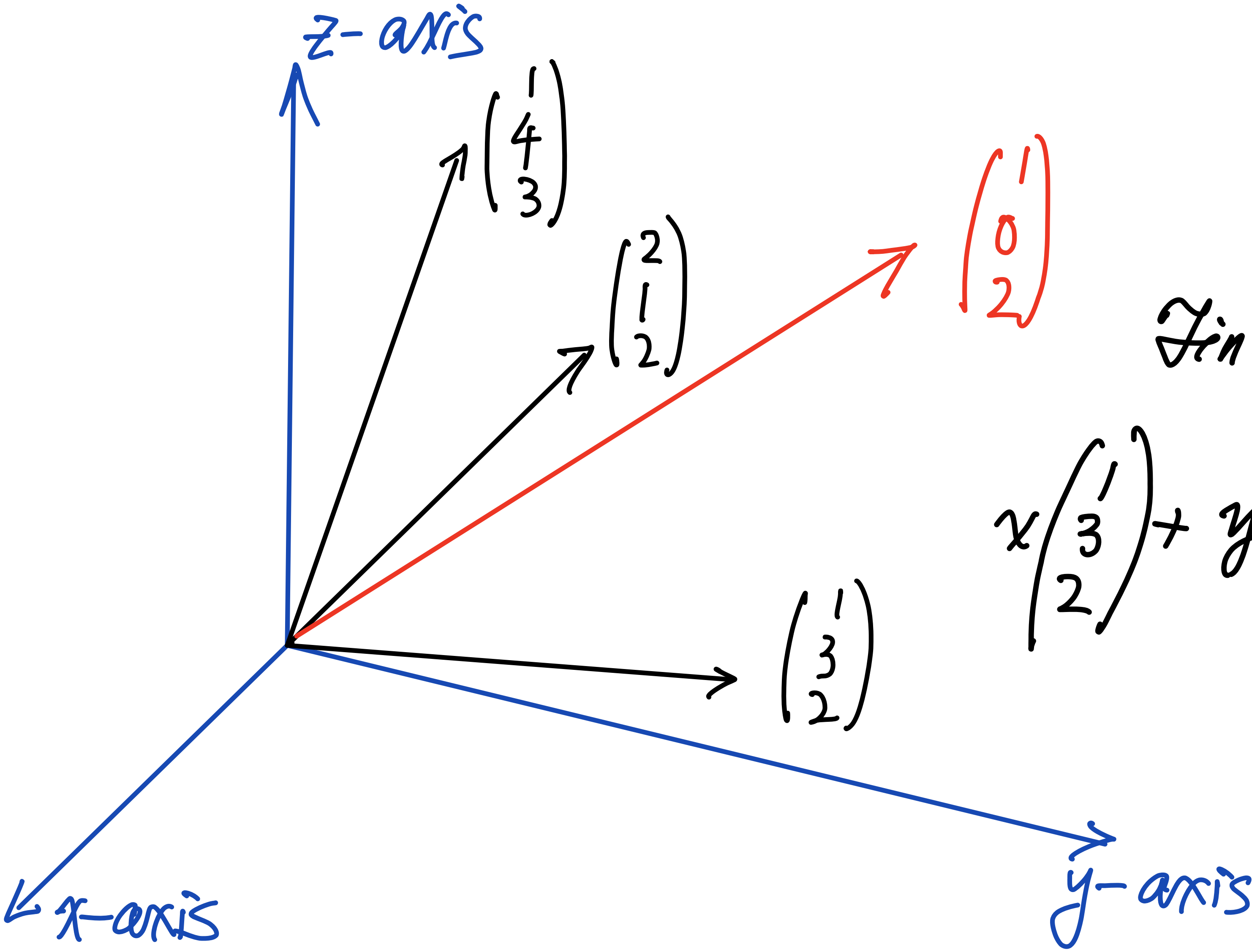
$$\Updownarrow$$

$$\begin{pmatrix} x \\ 3x \\ 2x \end{pmatrix} + \begin{pmatrix} 2y \\ y \\ 2y \end{pmatrix} + \begin{pmatrix} z \\ 4z \\ 3z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$$\Updownarrow$$

$$\boxed{x\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + y\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + z\begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}}$$

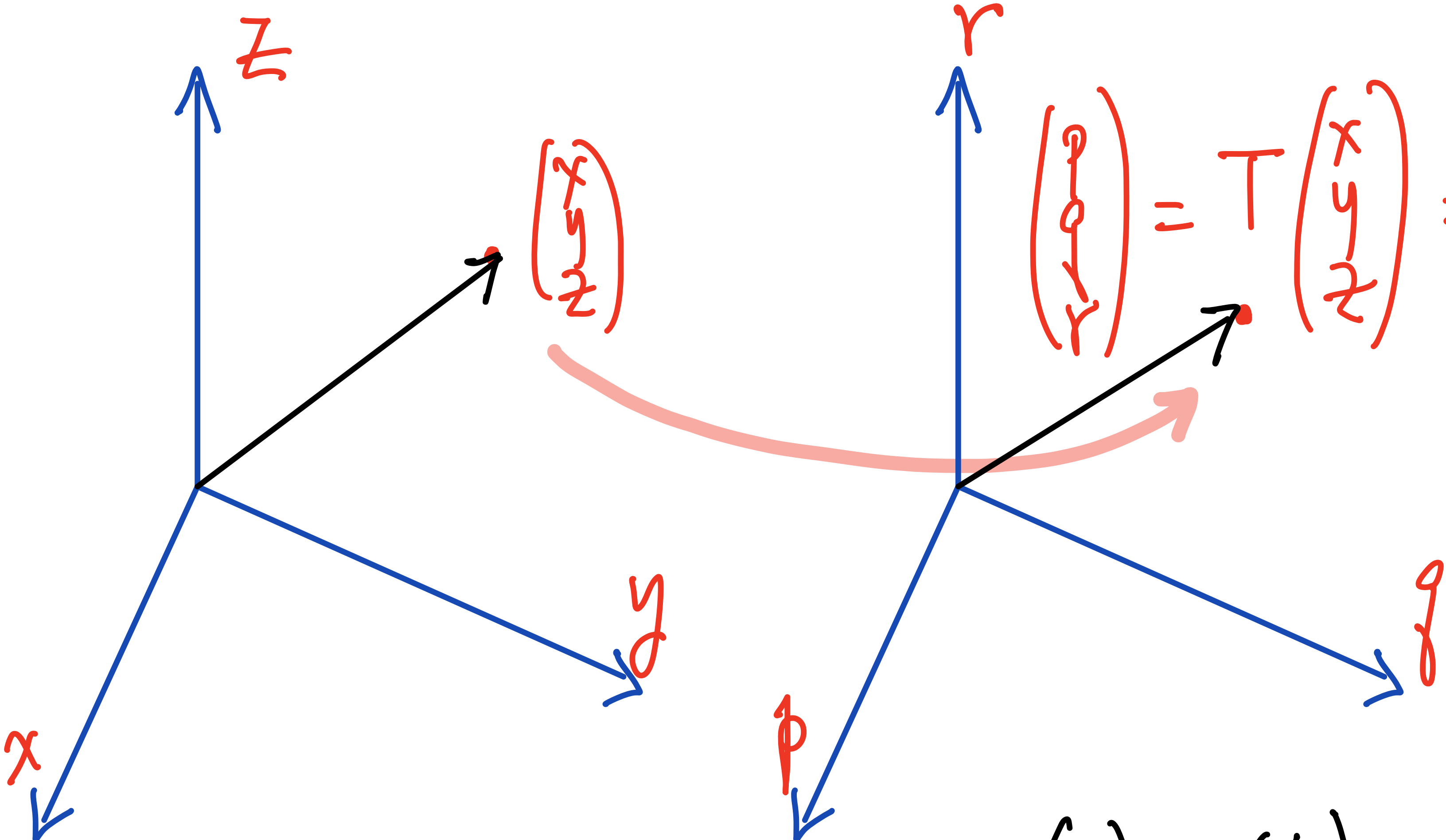
(II) Writing Vector as linear combination



Find x, y, z:

$$x\begin{pmatrix}1\\3\\2\end{pmatrix}+y\begin{pmatrix}2\\1\\2\end{pmatrix}+z\begin{pmatrix}1\\4\\3\end{pmatrix}=\begin{pmatrix}1\\0\\2\end{pmatrix}$$

(III) Linear Transformation



$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+2y+z \\ 3x+y+4z \\ 2x+2y+3z \end{pmatrix}$$

Find $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ such that $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

Solution of (*) (Using elimination & substitution)

$$\begin{cases} x + 2y + z = 1 & \text{—— (1)} \\ 3x + y + 4z = 0 & \text{—— (2)} \\ 2x + 2y + 3z = 2 & \text{—— (3)} \end{cases}$$

(Solution of Quiz #1)

$$(1) \implies x = 1 - 2y - z$$

$$(2) \implies 3(1-2y-z) + y + 4z = 0 \implies -5y + z = -3$$

$$(3) \implies 2(1-2y-z) + 2y + 3z = 2 \implies \underbrace{-2y + z = 0}$$

$$\begin{aligned} x &= 1 - 2(1) - 2 \\ &= -3 \end{aligned} \quad \Longleftarrow \quad \begin{aligned} y &= 1 \\ z &= 2 \end{aligned}$$

$$\boxed{\text{Solution: } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}}$$

"Just to make sure":

(I): $\begin{cases} (-3) + 2(1) + (2) = 1 \\ 3(-3) + (1) + 4(2) = 0 \\ 2(-3) + 2(1) + 3(2) = 2 \end{cases}$

(II): $-3\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + 1\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$