Gaussian Elimination – Solving Linear Systems

m equations in n unknowns $(m \times n)$

$$
\begin{cases}
a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n = b_1 & \text{—— } 1 \\
a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n = b_2 & \text{—— } 2 \\
a_{32}x_1 + a_{32}x_2 + \cdots\cdots + a_{3n}x_n = b_3 & \text{—— } 3 \\
\\
a_{i1}x_1 + \cdots + a_{ij}x_j + \cdots + a_{in}x_n = b_i & \text{—— } i \\
\qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots & \\
a_{m1}x_1 + a_{m2}x_2 + \cdots\cdots + a_{mn}x_n = b_m & \text{—— } m
\end{cases}
$$

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$$
\begin{cases}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 & \text{—— } (1) \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 & \text{—— } (2) \\
a_{32}x_1 + a_{32}x_2 + \cdots + a_{3n}x_1 = b_3 & \text{—— } (3) \\
\\
a_{i1}x_1 + \cdots + a_{ij}x_j + \cdots + a_{in}x_n = b_i & \text{—— } (i) \\
\vdots \qquad \vdots \qquad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m & \text{—— } (m)
\end{cases}
$$

a_{ij}

row no — i, j — column number

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m equations in n unknowns $(m \times n)$

$$
\underbrace{\left[\begin{array}{c|c}
\underbrace{\begin{matrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\
a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\
 & \vdots & \vdots & & \\
a_{m1} & a_{m2} & a_{m3} & & a_{mn}
\end{matrix}}_{\text{coeff. matrix}}
&
\begin{matrix}
b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n
\end{matrix}
\end{array}\right]}_{\text{augmented matrix}}
$$

Elementary Row Operations

(to eliminate variables)

(1) Interchange 2 rows (equations)

$$\begin{pmatrix} - & R_i & - \\ - & R_j & - \end{pmatrix} \longrightarrow \begin{pmatrix} - & R_j & - \\ - & R_i & - \end{pmatrix}$$

(2) Multiply 1 row (eqn) by a non-zero no.

$$\begin{pmatrix} - & R_i & - \end{pmatrix} \xrightarrow[(\alpha \neq 0)]{\times \alpha} \begin{pmatrix} - & \alpha R_i & - \end{pmatrix}$$

(3) Add a multiple of 1 row (eqn) to another

$$\begin{pmatrix} - & R_i & - \\ - & R_j & - \end{pmatrix} \xrightarrow{+\alpha R_i} \begin{pmatrix} - & R_i & - \\ - & \alpha R_i + R_j & - \end{pmatrix}$$

Elementary Row Operations

(to eliminate variables)

Note: elementary row op. can be reversed.

Hence: the new system is equivalent to the original one in the sense that they have the same set of solutions.

$$\boxed{\left[\ \text{System } 1\ \right] \rightleftarrows \left[\ \text{System } 2\ \right]}$$

Row Echelon Form (REF)

(to facilitate backward substitution)

① make this no. to be 1

② use it to make all the no.s below it to be zero

$$\begin{bmatrix} * & * & * & \cdots & * \\ * & * & & & * \\ * & * & & & * \\ * & & & & \vdots \\ * & & & & \vdots \\ * & & & & * \end{bmatrix}$$

Row Echelon Form (REF)

(to facilitate backward substitution)

x_1 is eliminated

$$
\begin{bmatrix}
1 & * & * & \cdots & * \\
0 & * & & & * \\
0 & * & & & * \\
\vdots & & & & \vdots \\
0 & & & & *
\end{bmatrix}
$$

$(m-1)\times(n-1)$ system. REPEAT!

Row Echelon Form (REF)

(to facilitate backward substitution)

$$\left[\begin{array}{ccccc|c} 1 & * & * & \cdots & & * \\ 0 & 1 & * & & & * \\ 0 & & 1 & * & & * \\ \vdots & & & \ddots & * & \vdots \\ & & 0\text{'s} & & \ddots & \\ 0 & & & & 1 & * \end{array}\right]$$

upper triangular matrix

Backward Substitution: Start from the last

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 1 \\ 0 & 1 & 7 & 5 & 0 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right]$$

Backward Substitution: Start from the last

$$
\begin{array}{cccccc}
x_1 & x_2 & x_3 & x_4 & & \\
\end{array}
$$

$$
\left[\begin{array}{cccc|c}
1 & 2 & -1 & 0 & 1 \\
0 & 1 & 7 & 5 & 0 \\
0 & 0 & 1 & 1 & 2 \\
0 & 0 & 0 & 1 & -3
\end{array}\right]
$$

1 $x_4 = -3$

2 $x_3 + x_4 = 2$

$x_3 = 5$

3 $x_2 + 7x_3 + 5x_4 = 0$

$x_2 = -20$

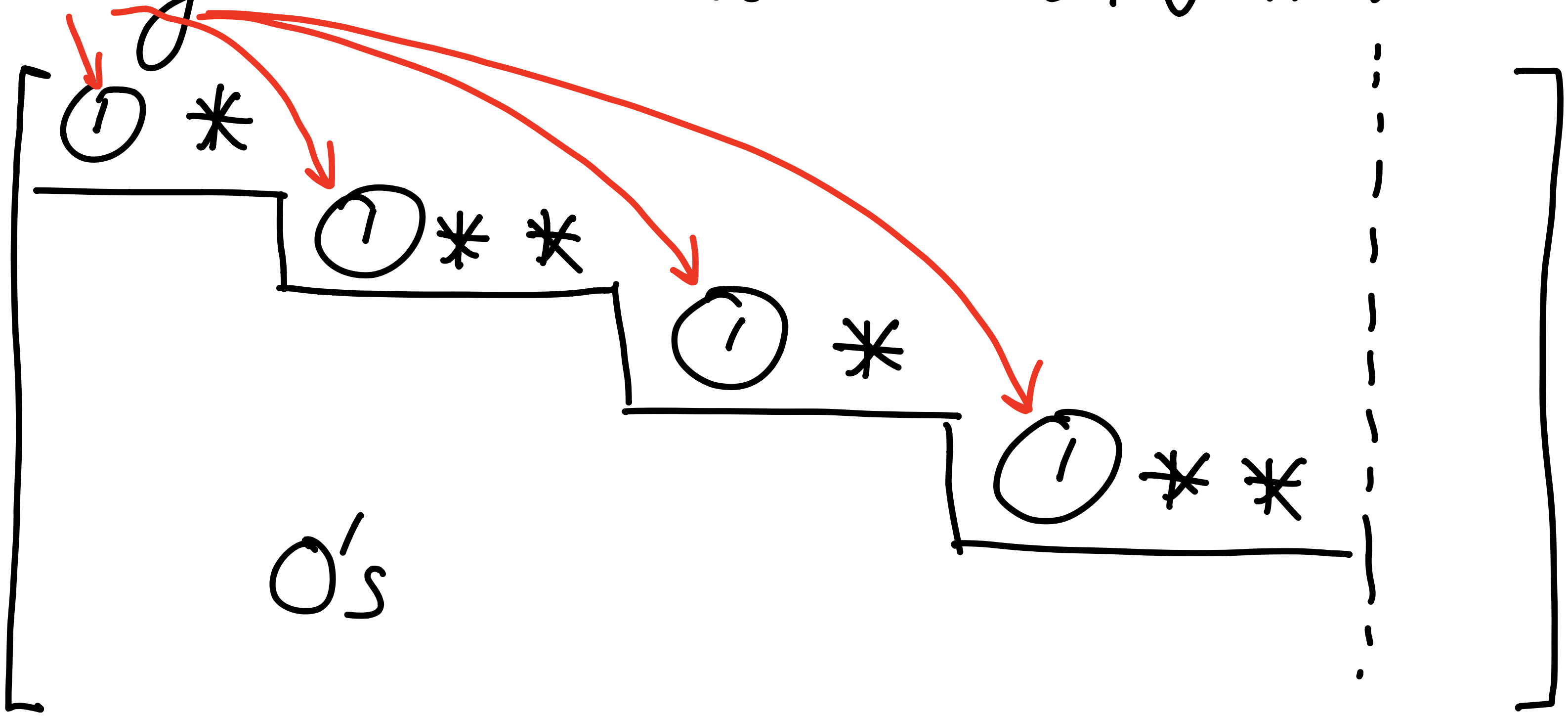
1 $x_1 + 2x_2 - x_3 = 1$

$x_1 = 46$

$$\boxed{\text{Solution: } \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 46 \\ -20 \\ 5 \\ -3 \end{pmatrix}}$$

Pivot vs Free Variables

It can happen that not all variables are "leading" in the row echelon form.



- The variables corresponding to the 1's in the REF are called pivot variables.
- The rest, non-leading, are called free variables

Examples: $\boxed{\text{Express pivot var. in terms of free var.}}$

(a)

(3x4) $\begin{cases} x+y+z+w=5 \\ x+y-z+w=7 \\ x+y+2z-w=0 \end{cases}$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 1 & 1 & -1 & 1 & 7 \\ 1 & 1 & 2 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & 0 & 1 & -2 & -5 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

(x, z, w are pivot, y is free)

Examples: $\boxed{\text{Express pivot var. in terms of free var.}}$

(a) $\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & 0 & 1 & -2 & -5 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$

$w = 2$

$z - 2w = -5 \Rightarrow z = -1$

$y = t$ (free)

$$\begin{aligned} x &= -y - z - w + 5 \\ &= -t + 1 - 2 + 5 \\ &= 4 - t \end{aligned}$$

Solution:

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 4-t \\ t \\ -1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ 0 \\ -1 \\ 2 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Examples: $\boxed{\text{Express pivot var. in terms of free var.}}$

(b) $x + 2y + 5z + 3w = 0$

$z - w = 7$

(2×4)

$$\left[\begin{array}{cccc|c} 1 & 2 & 5 & 3 & 0 \\ 0 & 0 & 1 & -1 & 7 \end{array}\right]$$

(x, z are pivot
y, w are free)

$w = t$ (free)

$z = w + 7 = t + 7$

$y = s$ (free)

$x = -2y - 5z - 3w$
$= -2s - 5(t+7) - 3t$
$= -35 - 2s - 8t$

Solution

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -35-2s-8t \\ s \\ t+7 \\ t \end{pmatrix} = \begin{pmatrix} -35 \\ 0 \\ 7 \\ 0 \end{pmatrix} + s\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -8 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Examples: $\boxed{\text{Express pivot var. in terms of free var.}}$

(c) $x - y + z = 5$ (1x3)

$[1 \;\; -1 \;\; 1 \mid 5]$

(x is pivot, y, z are free)

$z = t$ (free)

$y = s$ (free)

$x = y - z + 5$

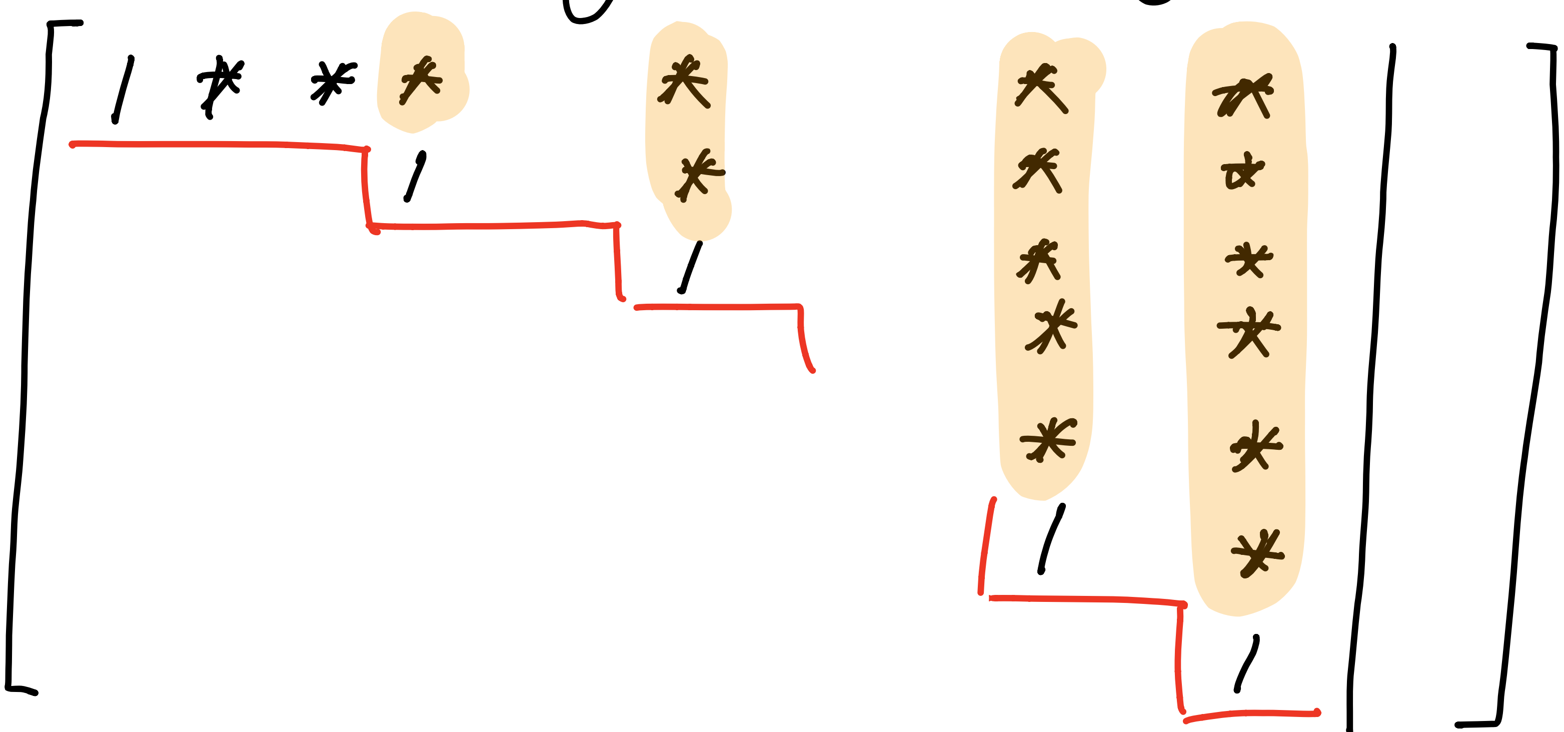
$= s - t + 5$

Solution:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s - t + 5 \\ s \\ t \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Reduced Row Echelon Form (RREF)

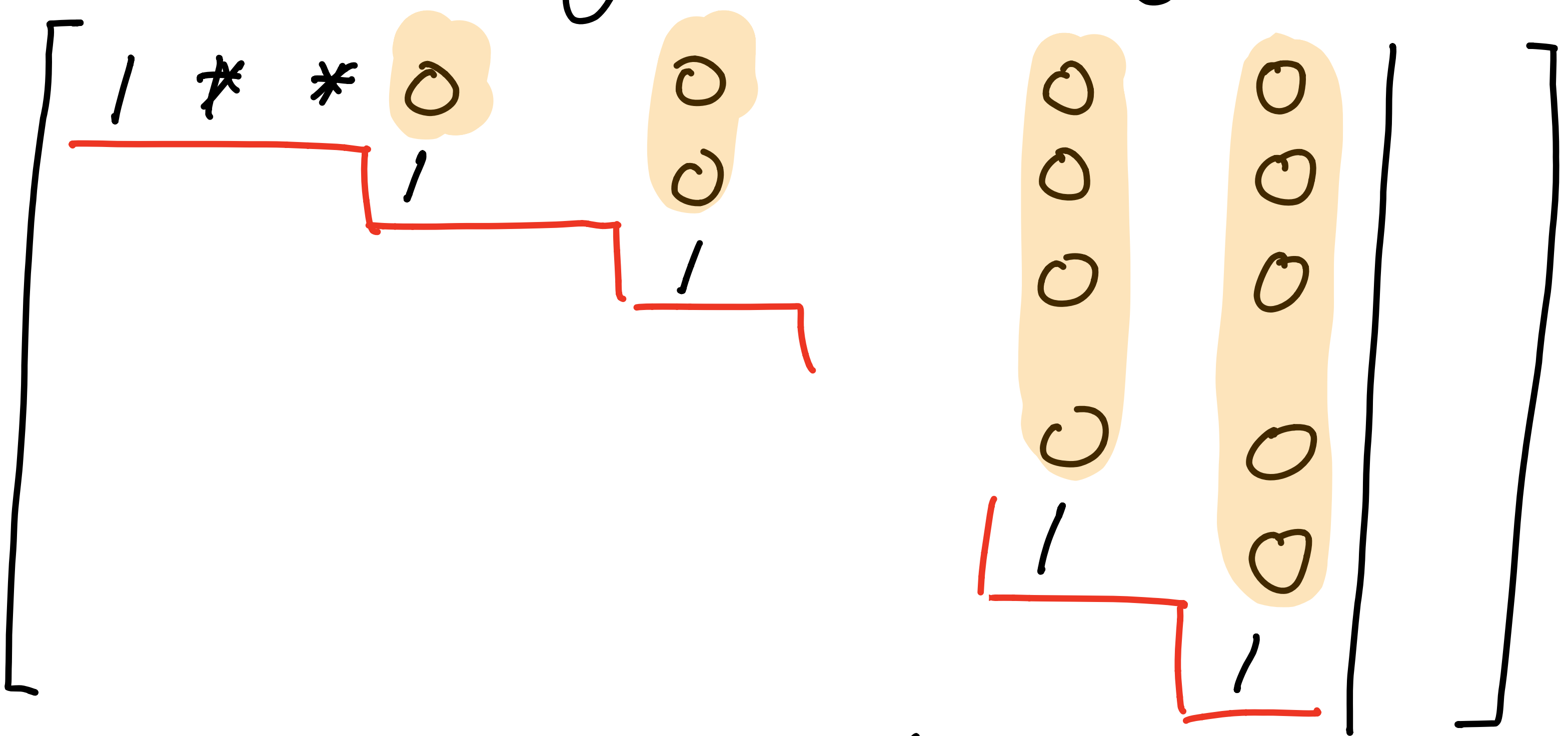
(Backward substitution can also be achieved using elementary row ops.)



Use the pivot 1's to make all the numbers above them to be zero (start from the last row)

Reduced Row Echelon Form (RREF)

(Backward substitution can also be achieved using elementary row ops.)



Use the pivot 1's to make all the numbers above them to be zero (start from the last row)

Three Scenarios of Solving Linear Systems

(1) **Unique Solution** (only pivot variables, i.e. no free variables, and in REF, no. of non-zero rows in the coefficient matrix is the same as the no. of non-zero rows in the augmented matrix.)

$$\left[\begin{array}{ccccccc|c}
1 & * & * & & & & & * \\
 & 1 & * & & & & & * \\
 & & 1 & * & & & & * \\
 & & & 1 & & & & * \\
 & & & & 1 & & & * \\
 & \text{0's} & & & & 1 & * & * \\
 & & & & & & 1 & *
\end{array}\right]$$

Three Scenarios of Solving Linear Systems

(2) **Infinitely many solutions** (There are free variables, and in REF, no. of non-zero rows in the coefficient matrix is the same as the no. of non-zero rows in the augmented matrix.)

$$
\left[\begin{array}{ccccccccccccc|c}
1 & * & & & & & & & & & & & & * \\
 & 1 & * & * & & & & & & & & & & * \\
 & & & & 1 & & & & & & & & & * \\
 & & & & & 1 & * & * & & & & & & * \\
 & \text{0's} & & & & & & & 1 & * & & & & * \\
 & & & & & & & & & & 1 & * & * & *
\end{array}\right]
$$

Three Scenarios of Solving Linear Systems

(3) No Solution (inconsistent)

in REF, no. of non-zero rows in the coefficient matrix is less than the no. of non-zero rows in the augmented matrix.)

no. of non-zero rows in the coefficient matrix

$$\left[\begin{array}{ccccc|c} 1 & & & & & * \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & 0's & & & 1 & * \\ & & & & 0 & * \\ & & & & 0 & * \end{array}\right]$$

no. of non-zero rows in the augmented matrix