

Gaussian Elimination - Solving linear Systems

m equations in n unknowns ($m \times n$)

$$\left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \quad — ① \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \quad — ② \\ a_{31}x_1 + a_{32}x_2 + \dots + a_{3n}x_n = b_3 \quad — ③ \\ \vdots \qquad \vdots \qquad \vdots \\ a_{i1}x_1 + \dots + a_{ij}x_j + \dots + a_{in}x_n = b_i \quad — i \\ \vdots \qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \quad — m \end{array} \right.$$

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$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \quad \text{--- } ①$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \quad \text{--- } ②$$

$$a_{31}x_1 + a_{32}x_2 + \dots + a_{3n}x_n = b_3 \quad \text{--- } ③$$

$$a_{i1}x_1 + \dots + a_{ij}x_j + \dots + a_{in}x_n = b_i \quad \text{--- } i$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \quad \text{--- } m$$

a_{ij}
row no \rightarrow column number

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m equations in n unknowns ($m \times n$)

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ \vdots & \vdots & \vdots & & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_n \end{array} \right]$$

coeff. matrix

augmented matrix

Elementary Row Operations

(to eliminate variables)

(1) Interchange 2 rows (equations)

$$\begin{array}{c} (\text{--- } R_i \text{ ---}) \\ (\text{--- } R_j \text{ ---}) \end{array} \xrightarrow{\hspace{1cm}} \begin{array}{c} (\text{--- } R_j \text{ ---}) \\ (\text{--- } R_i \text{ ---}) \end{array}$$

(2) Multiply 1 row (eqn) by a non-zero no.

$$(\text{--- } R_i \text{ ---}) \xrightarrow[\substack{(\alpha \neq 0)}]{\times \alpha} (\text{--- } \alpha R_i \text{ ---})$$

(3) Add a multiple of 1 row (eqn) to another

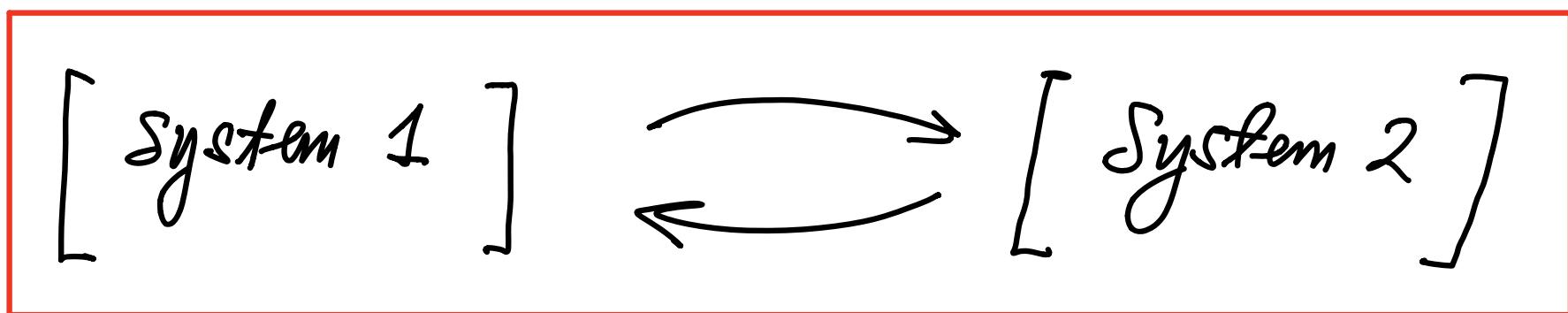
$$\begin{array}{c} (\text{--- } R_i \text{ ---}) \\ (\text{--- } R_j \text{ ---}) \end{array} \xrightarrow{+ \alpha R_i} \begin{array}{c} (\text{--- } R_i \text{ ---}) \\ (\text{--- } \alpha R_i + R_j \text{ ---}) \end{array}$$

Elementary Row Operations

(to eliminate variables)

Note: elementary row op. can be reversed.

Hence: the new system is equivalent to the original one in the sense that they have the same set of solutions.



Row Echelon Form (REF)

(to facilitate backward substitution)

- ① make this no. to be 1
- ② use it to make all the no.s below it to be zero

The diagram illustrates the Row Echelon Form (REF) of a matrix. On the left, a vertical column of six asterisks is highlighted with a yellow bar, and the top asterisk is circled in red with an arrow pointing to it from the text "make this no. to be 1". A second arrow points from the text "use it to make all the no.s below it to be zero" to the same yellow-highlighted column. To the right of this column, the matrix is shown in a REF format: the first row has one asterisk circled in red; the second row has two asterisks circled in red; the third row has three asterisks circled in red; the fourth row has four asterisks circled in red; and the fifth row has five asterisks circled in red. Dashed lines indicate that there are more rows below the fifth row. To the far right, a vertical bracket groups the last four columns of the matrix.

$$\left[\begin{array}{ccccc|c} * & * & * & - & - & * \\ * & * & * & & & * \\ * & * & * & & & * \\ * & * & * & & & * \\ * & * & * & & & * \end{array} \right]$$

Row Echelon Form (REF)

(to facilitate backward substitution)

X_1 is eliminated

The diagram illustrates a matrix in Row Echelon Form (REF). The matrix has m rows and n columns. The first column contains entries 1, 0, 0, \vdots , 0. The second column contains entries *, 0, *, 0, \vdots , *. The third column contains entries *, 0, *, 0, \vdots , *. The fourth column contains entries 0, 0, 0, \vdots , 0. Red dashed lines indicate row operations: one line connects the first two rows, another connects the second and third rows, and a bracket at the bottom connects the last $n-1$ rows. A red circle highlights the leading coefficient 1 in the first row. A red arrow points from the text "X₁ is eliminated" to the first column.

$(m-1) \times (n-1)$ system. REPEAT!

Row Echelon Form (REF)

(to facilitate backward substitution)

The diagram illustrates the Row Echelon Form (REF) of a matrix. On the left, a vertical line separates the matrix from the right-hand side. The matrix itself has several circled entries: the first entry in the first row is circled in red; the second entry in the second row is circled in red; and the third entry in the third row is circled in red. The matrix contains a diagonal line of asterisks (*). To the left of the matrix, there is a vertical ellipsis (three dots) and a horizontal ellipsis (- - - - -) above the diagonal. Below the matrix, the text "O's" is written next to a wavy line that points towards the bottom-left corner of the matrix. A red arrow points from the bottom-left corner of the matrix towards the left edge, with the number "0" written below it. To the right of the vertical line, there is another vertical line enclosing a column of asterisks (*).

upper triangular matrix

Backward Substitution: Start from the last

$$\left[\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ \hline 1 & 2 & -1 & 0 & 1 \\ 0 & 1 & 7 & 5 & 0 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & -3 \end{array} \right]$$

Backward Substitution: Start from the last

$$\left[\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ \hline 1 & 2 & -1 & 0 & 1 \\ 0 & 1 & 7 & 5 & 0 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & -3 \end{array} \right]$$

① $x_1 + 2x_2 - x_3 = 1$
 $x_1 = 46$

② $x_3 + x_4 = 2$
 $x_3 = 5$

③ $x_2 + 7x_3 + 3x_4 = 0$
 $x_2 = -20$

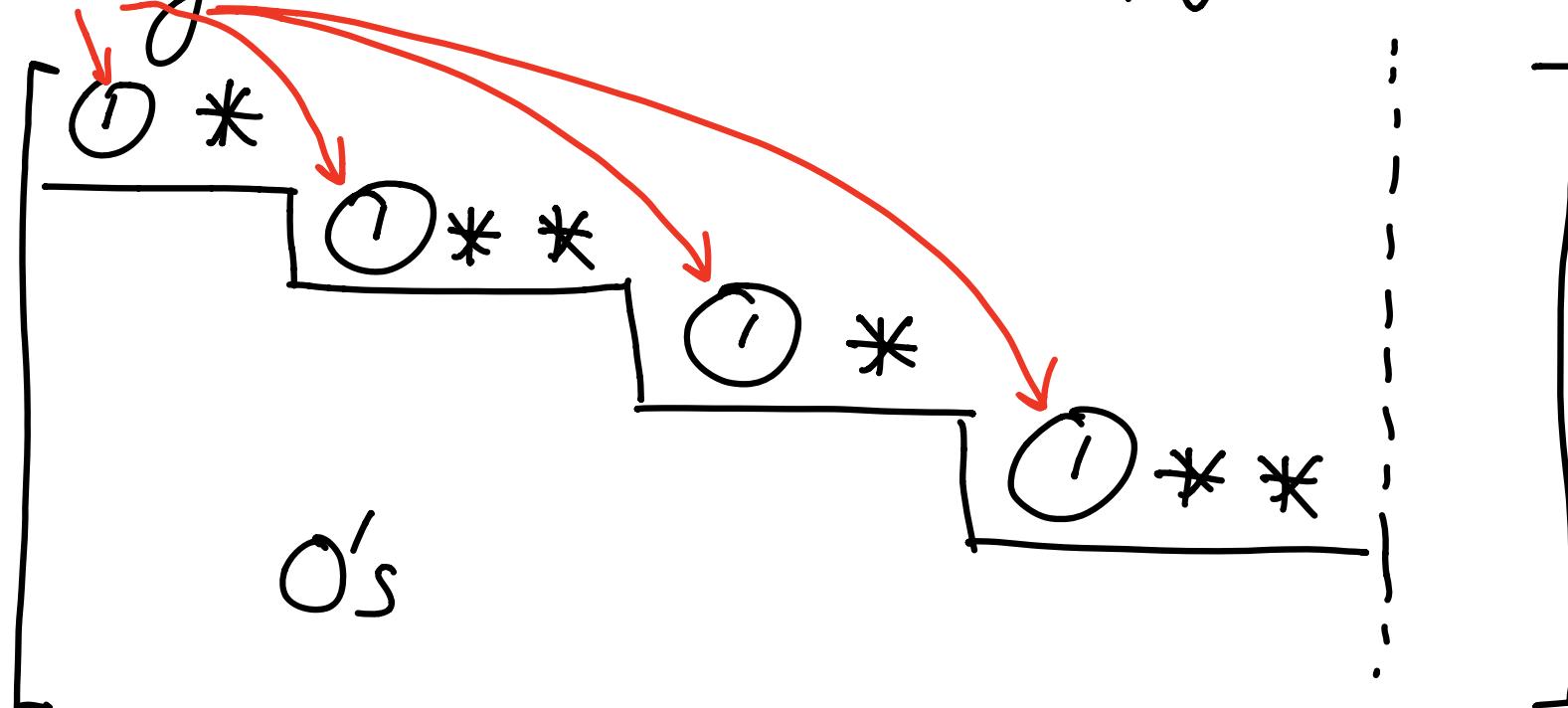
④ $x_4 = -3$

Solution:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 46 \\ -20 \\ 5 \\ -3 \end{pmatrix}$$

Pivot vs Free Variables

It can happen that not all variables are "leading" in the row echelon form.



- The variables corresponding to the 1's in the REF are called pivot variables.
- The rest, non-leading, are called free variables

Examples: Express pivot var. in terms of free var.

(a)

$$(3 \times 4) \left\{ \begin{array}{l} x + y + z + w = 5 \\ x + y - z + w = 7 \\ x + y + 2z - w = 0 \end{array} \right.$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 1 & 1 & -1 & 1 & 7 \\ 1 & 1 & 2 & -1 & 0 \end{array} \right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & 0 & 1 & -2 & -5 \\ 0 & 0 & 0 & 1 & 2 \end{array} \right]$$

(x, z, w are pivot,
 y is free)

Examples: Express pivot var. in terms of free var.

(a)

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & 0 & 1 & -2 & -5 \\ 0 & 0 & 0 & 1 & 2 \end{array} \right]$$

$$w = 2$$

$$z - 2w = -5 \Rightarrow z = -1$$

$$y = t \text{ (free)}$$

$$x = -y - z - w + 5$$

$$= -t + 1 - 2 + 5$$

$$= 4 - t$$

Solution:

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 4-t \\ t \\ -1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ 0 \\ -1 \\ 2 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Examples: Express pivot var. in terms of free var.

$$(5) \begin{array}{l} x + 2y + 5z + 3w = 0 \\ z - w = 7 \end{array} \quad (2 \times 4)$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 5 & 3 & 0 \\ 0 & 0 & 1 & -1 & 7 \end{array} \right] \quad \begin{array}{l} (x, z \text{ are pivot}) \\ (y, w \text{ are free}) \end{array}$$

$\downarrow \quad \downarrow \quad \downarrow$

$w = t \text{ (free)}$

$$z = w + 7 = t + 7$$

$$y = s \text{ (free)}$$

$$x = -2y - 5z - 3w$$

$$= -2s - 5(t+7) - 3t$$

$$= -35 - 2s - 8t$$

Solution

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -35 - 2s - 8t \\ s \\ t + 7 \\ t \end{pmatrix} = \begin{pmatrix} -35 \\ 0 \\ 7 \\ 0 \end{pmatrix} + s \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 8 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

Examples: Express pivot var. in terms of free var.

(c) $x - y + z = 5$ (1x3)

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \end{array} \right]$$

(x is pivot,
 y, z are free)

$$y = s \text{ (free)}$$



$$x = y - z + 5$$

$$= s - t + 5$$

Solution:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s-t+5 \\ s \\ t \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Reduced Row Echelon Form (RREF)

(Backward substitution can also be achieved using elementary row ops.)

A hand-drawn diagram illustrating a memory or association task. It consists of two main sections separated by a horizontal line.

Left Section:

- The first column contains a single character: 犬 (Dog).
- The second column contains two characters: 犬 (Dog) and 米 (Rice).
- The third column contains three characters: 犬 (Dog), 米 (Rice), and 米 (Rice).

Right Section:

- The first column contains four characters: 犬 (Dog), 犬 (Dog), 米 (Rice), and 米 (Rice).
- The second column contains five characters: 犬 (Dog), 犬 (Dog), 米 (Rice), 米 (Rice), and 米 (Rice).

Red lines connect the first character of each row in the left section to the first character of each row in the right section, indicating a one-to-one correspondence between the first elements of each group.

Use the pivot 1's to make all the numbers above them to be zero. (start from the last row)

Reduced Row Echelon Form (RREF)

Backward substitution can also be achieved using elementary row ops.)

$$\left[\begin{array}{cccc|c} 1 & * & * & 0 & 0 \\ & 1 & & & 0 \\ & & 1 & & 0 \\ & & & 1 & 0 \\ & & & & 1 \\ & & & & & 1 \\ & & & & & & 1 \end{array} \right]$$

Use the pivot 1's to make all the numbers above them to be zero. (start from the last row)

Three Scenarios of Solving Linear Systems

(1) Unique Solution

(only pivot variables,
i.e. no free variables, and in REF, no. of non-zero
rows in the coefficient matrix is the same as the no.
of non-zero rows in the augmented matrix.)

$$\left[\begin{array}{ccc|c} 1 & * & * & * \\ 1 & * & & * \\ 1 & * & 1 & * \\ 0's & & & 1 \\ \end{array} \right]$$

Three Scenarios of Solving Linear Systems

(2) Infinitely many solutions (There are free variables, and in REF, no. of non-zero rows in the coefficient matrix is the same as the no. of non-zero rows in the augmented matrix.)

$$\left[\begin{array}{cc|c} 1 & * & * \\ | & * & * \\ | & * & * \\ \text{O's} & & | \\ & & | & * & * \\ & & | & * & * \\ & & | & * & * \end{array} \right]$$

Three Scenarios of Solving Linear Systems

(3) No Solution (inconsistent)

in REF, no. of non-zero rows in the coefficient matrix is less than the no. of non-zero rows in the augmented matrix.)

$$\left[\begin{array}{cccc|c} 1 & & & & & * \\ & 1 & & & & * \\ & & 1 & & & * \\ & & & 1 & & * \\ & & & & 1 & * \\ & & & & & 0 \\ & & & & & 0 \\ & & & & & 0 \end{array} \right]$$

no. of non-zero rows in the coefficient matrix } }

no. of non-zero rows in the augmented matrix }