Solving mxn Linear System

Note Title 2/3/2016

mxn linear system: (m equations in n unknowns)

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \quad \vdots \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

$\Updownarrow$ In matrix notation: $A\vec{X} = \vec{B}$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$\Updownarrow$ In terms of linear combination: (columns of A)

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

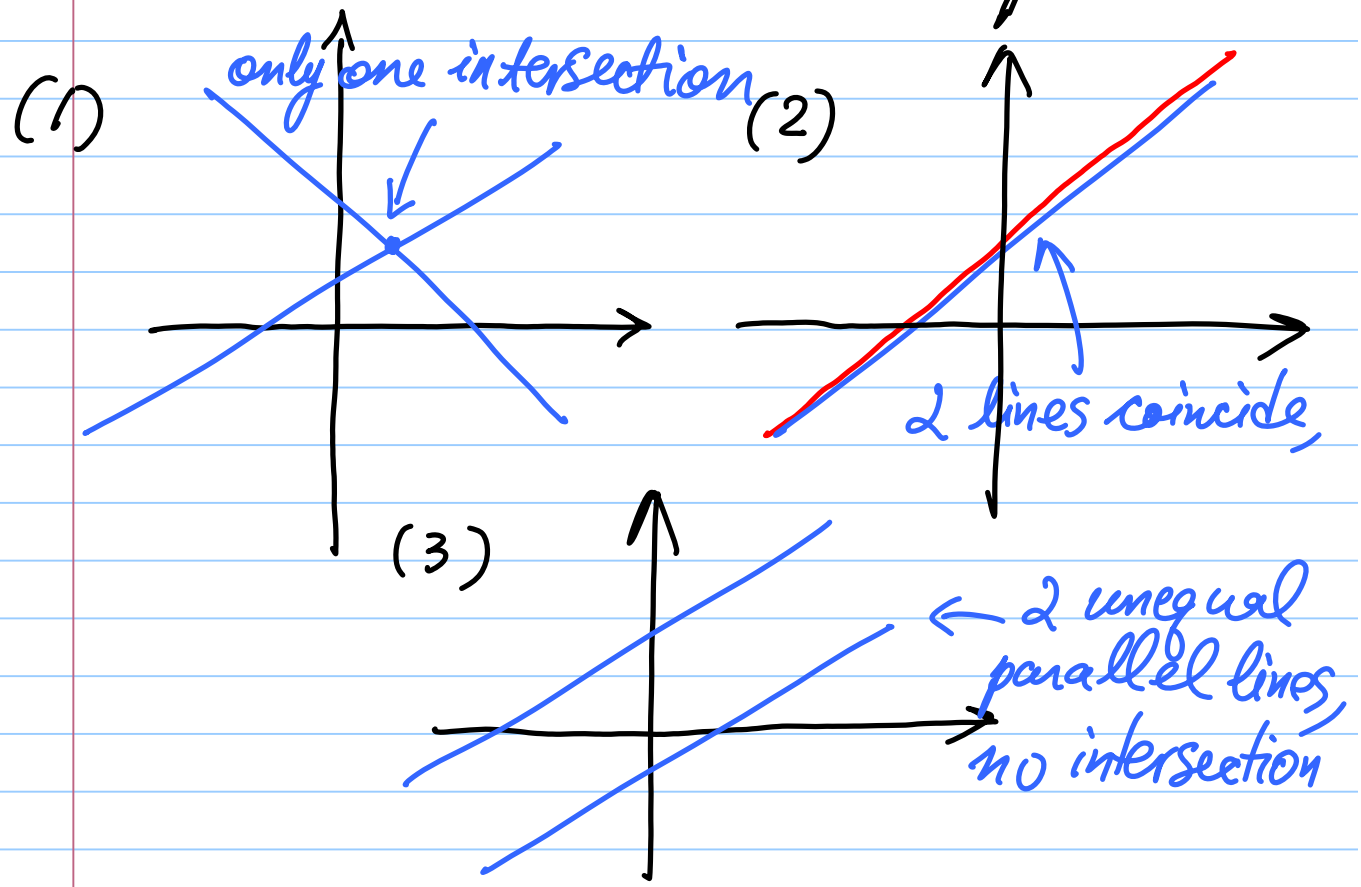
For linear system, one and only one of the following cases can occur:

(1) there is a unique (ie. only one) solution;

(2) there are infinitely many solutions;

(3) there is no solution.

In the case of 2X2 linear system, the above 3 cases correspond to:



Ex1 $\begin{cases} 2x+y=-1 \\ x-3y=5 \end{cases}$

$$\left(\begin{array}{cc|c} 2 & 1 & -1 \\ 1 & -3 & 5 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -3 & 5 \\ 2 & 1 & -1 \end{array}\right)$$

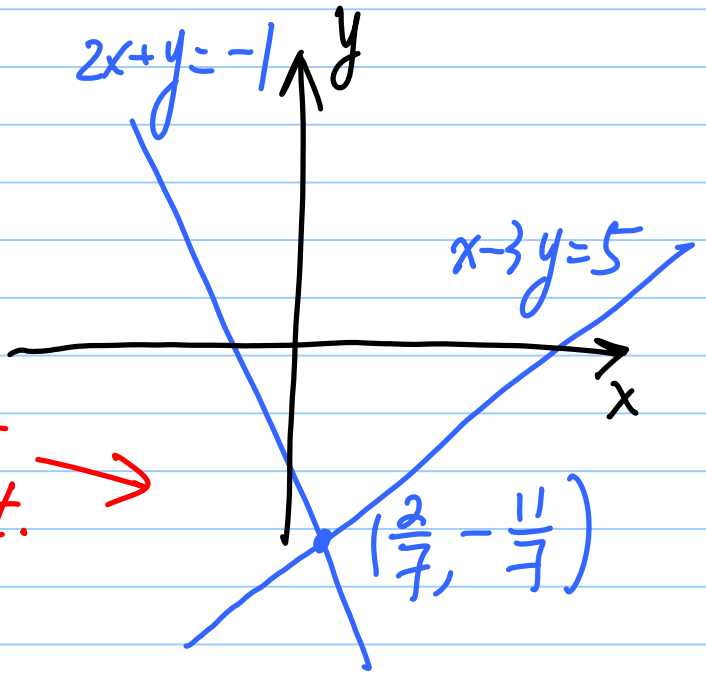
$$\to \left(\begin{array}{cc|c} 1 & -3 & 5 \\ 0 & 7 & -11 \end{array}\right)$$

$7y=-11 \Rightarrow y=-\frac{11}{7}$

$x-3y=5$

$x=5+3y=5-\frac{33}{7}=\frac{2}{7}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2/7 \\ -11/7 \end{pmatrix}$$



2 lines intersect at a unique point. →

Ex2

$$\begin{cases} x+2y+z=1 \\ 3x+y+4z=0 \\ 2x+2y+3z=2 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 3 & 1 & 4 & 0 \\ 2 & 2 & 3 & 2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -5 & 1 & -3 \\ 0 & -2 & 1 & 0 \end{array}\right)$$

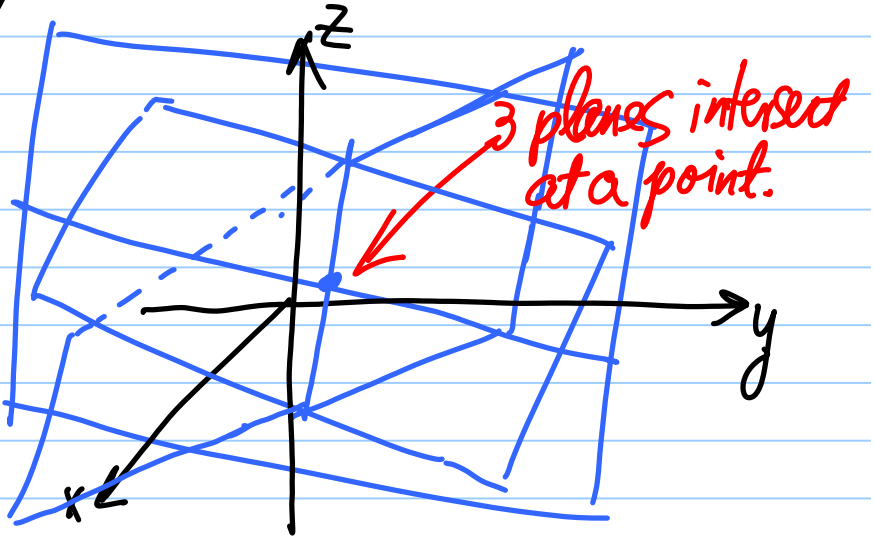
$$\to \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & -\frac{1}{5} & \frac{3}{5} \\ 0 & 1 & -\frac{1}{2} & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & -\frac{1}{5} & \frac{3}{5} \\ 0 & 0 & -\frac{3}{10} & -\frac{3}{5} \end{array}\right)$$

$$z = 2$$

$$y = \frac{3}{5} + \frac{z}{5} = 1$$

$$x = 1-2y-z = 1-2-2 = -3$$

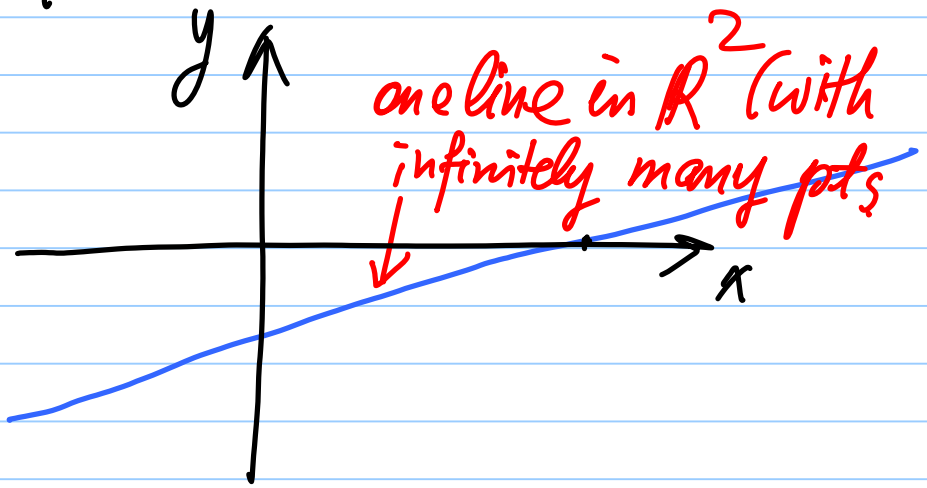
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$$



Ex3 $x-6y=7$

$y=s$ (free), $x=7+6s$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7+6s \\ s \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \end{pmatrix} + s\begin{pmatrix} 6 \\ 1 \end{pmatrix}$$



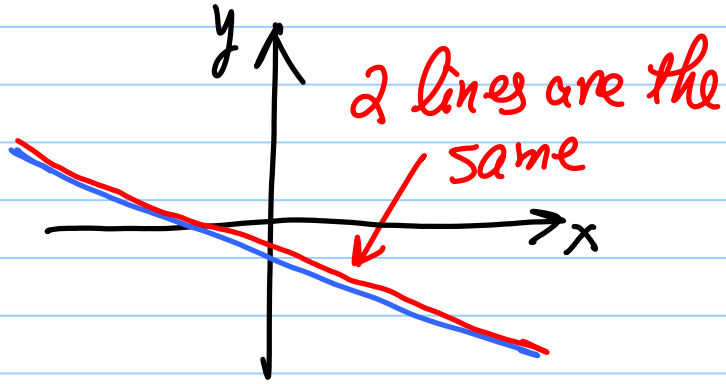
Ex4 $\begin{cases} x+3y=-1 \\ 4x+12y=-4 \end{cases}$

$$\begin{pmatrix} 1 & 3 & -1 \\ 4 & 12 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$y=s$ (free)

$x=-1+3s$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

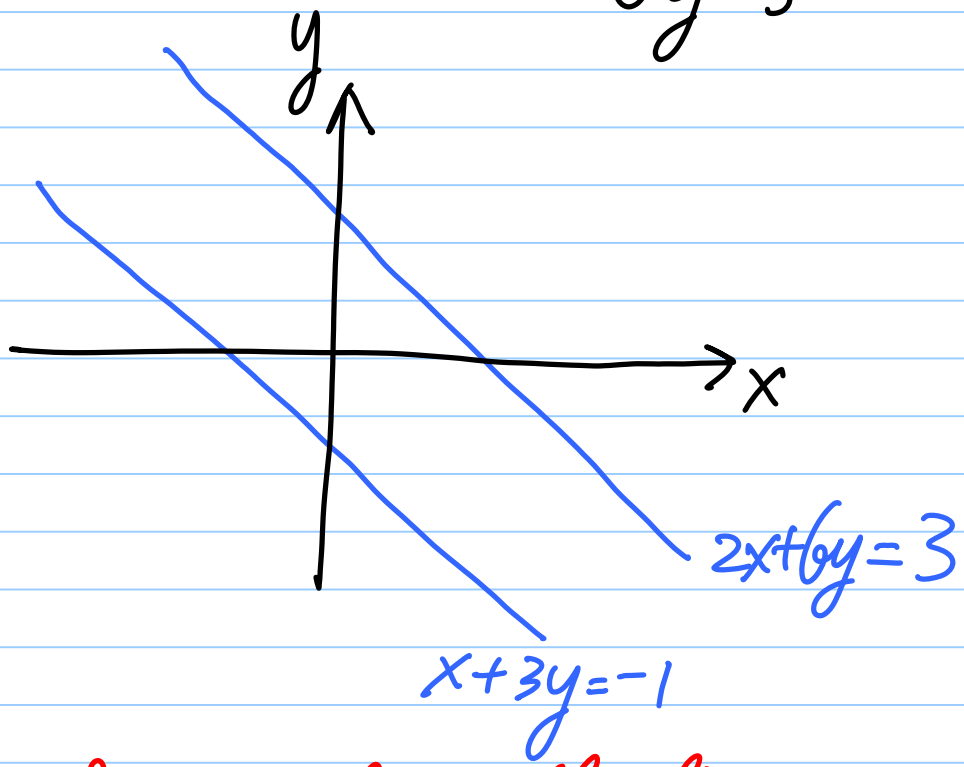


Ex5 $\begin{cases} x+3y=-1 \\ 2x+6y=3 \end{cases}$

$$\left(\begin{array}{cc|c} 1 & 3 & -1 \\ 2 & 6 & 3 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 3 & -1 \\ 0 & 0 & 5 \end{array}\right)$$

no solution

$0y=5$



2 unequal parallel lines,
no intersection

Ex6

$$\begin{cases} x+3y-z=5 \\ 2x-y+z=10 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 3 & -1 & 5 \\ 2 & -1 & 1 & 10 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 3 & -1 & 5 \\ 0 & -7 & 3 & 0 \end{array}\right)$$

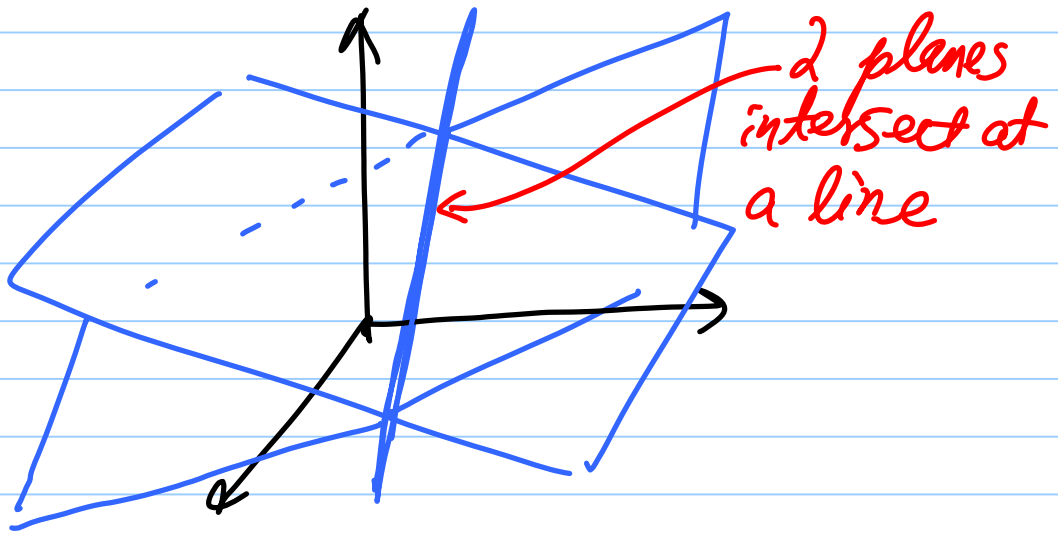
$$\to \left(\begin{array}{ccc|c} 1 & 3 & -1 & 5 \\ 0 & 1 & -3/7 & 0 \end{array}\right)$$

$z = \alpha$ (free)

$$y = \frac{3}{7}z = \frac{3}{7}\alpha$$

$$x = 5 - 3y + z = 5 - \frac{9}{7}\alpha + \alpha = 5 - \frac{2}{7}\alpha$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 - \frac{2}{7}\alpha \\ \frac{3}{7}\alpha \\ \alpha \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -2/7 \\ 3/7 \\ 1 \end{pmatrix}$$



Ex 7

$$\begin{cases} x - y + 2z = 7 \\ 2x - 2y + 4z = 14 \end{cases}$$

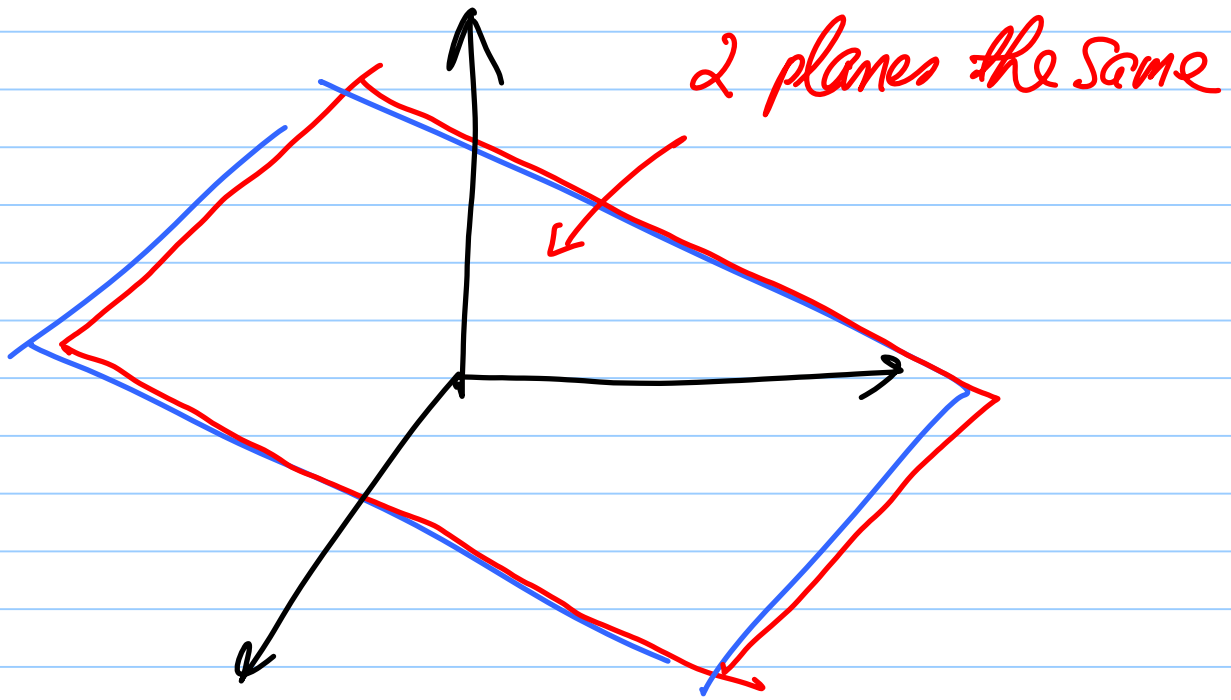
$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 7 \\ 2 & -2 & 4 & 14 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 2 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$z = \alpha$ (free)

$y = \beta$ (free)

$$x = \beta - 2\alpha + 7$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 + \beta - 2\alpha \\ \beta \\ \alpha \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

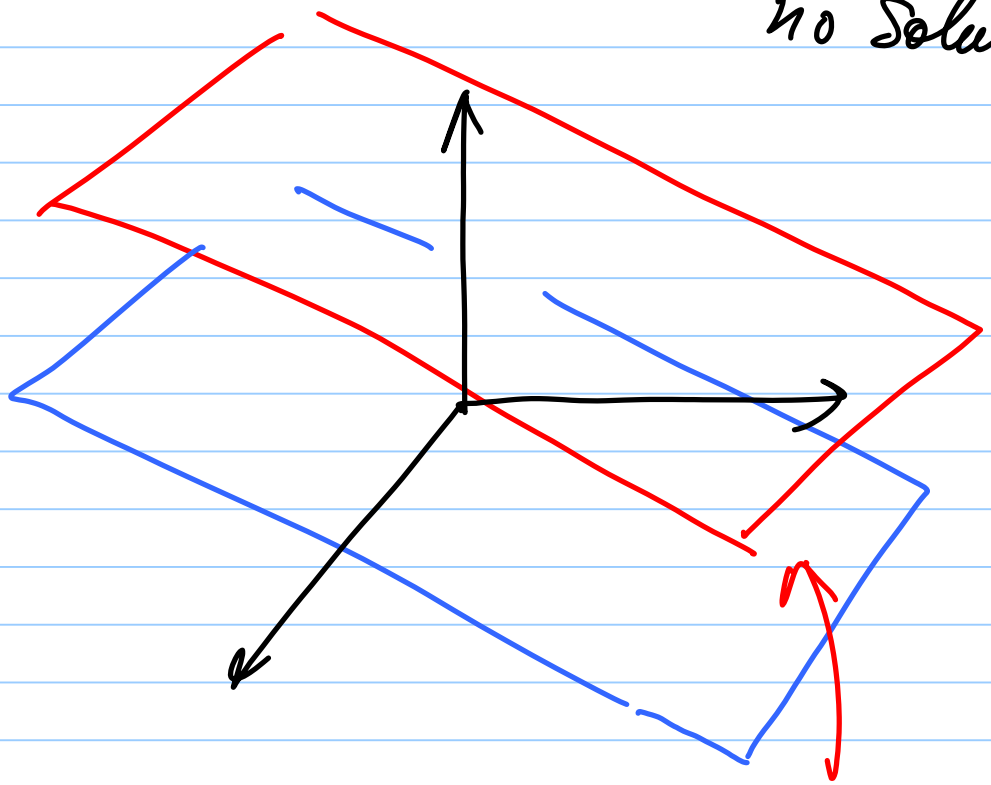


Ex 8

$$\begin{cases} x - y + 2z = 7 \\ 3x - 3y + 6z = 2 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 7 \\ 3 & -3 & 6 & 2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 2 & 7 \\ 0 & 0 & 0 & -19 \end{array}\right)$$

↗ no solution



2 unequal, parallel planes, no intersection

Ex 9

$$\begin{cases} x+y-z=2 \\ 3x-y+2z=0 \\ 2x-2y+3z=-2 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 3 & -1 & 2 & 0 \\ 2 & -2 & 3 & -2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & -4 & 5 & -6 \\ 0 & -4 & 5 & -6 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & -4 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & -5/4 & 3/2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

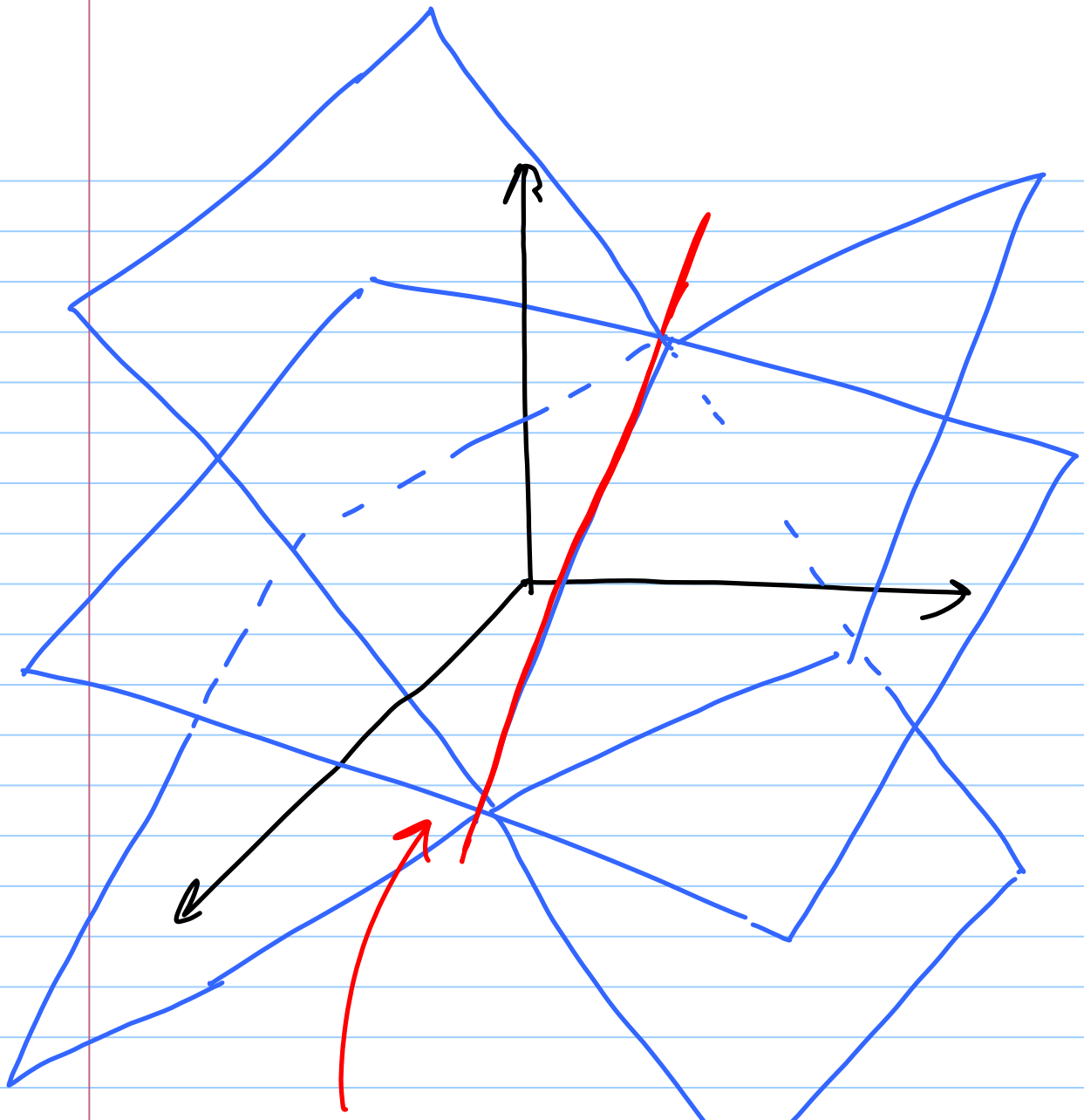
$z = \alpha$ (free)

$y = \frac{3}{2} + \frac{5}{4}\alpha$

$x = 2 - y + z = 2 - \frac{3}{2} - \frac{5}{4}\alpha + \alpha$

$= \frac{1}{2} - \frac{1}{4}\alpha$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2} - \frac{1}{4}\alpha \\ \frac{3}{2} + \frac{5}{4}\alpha \\ \alpha \end{pmatrix} = \begin{pmatrix} 1/2 \\ 3/2 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -1/4 \\ 5/4 \\ 1 \end{pmatrix}$$



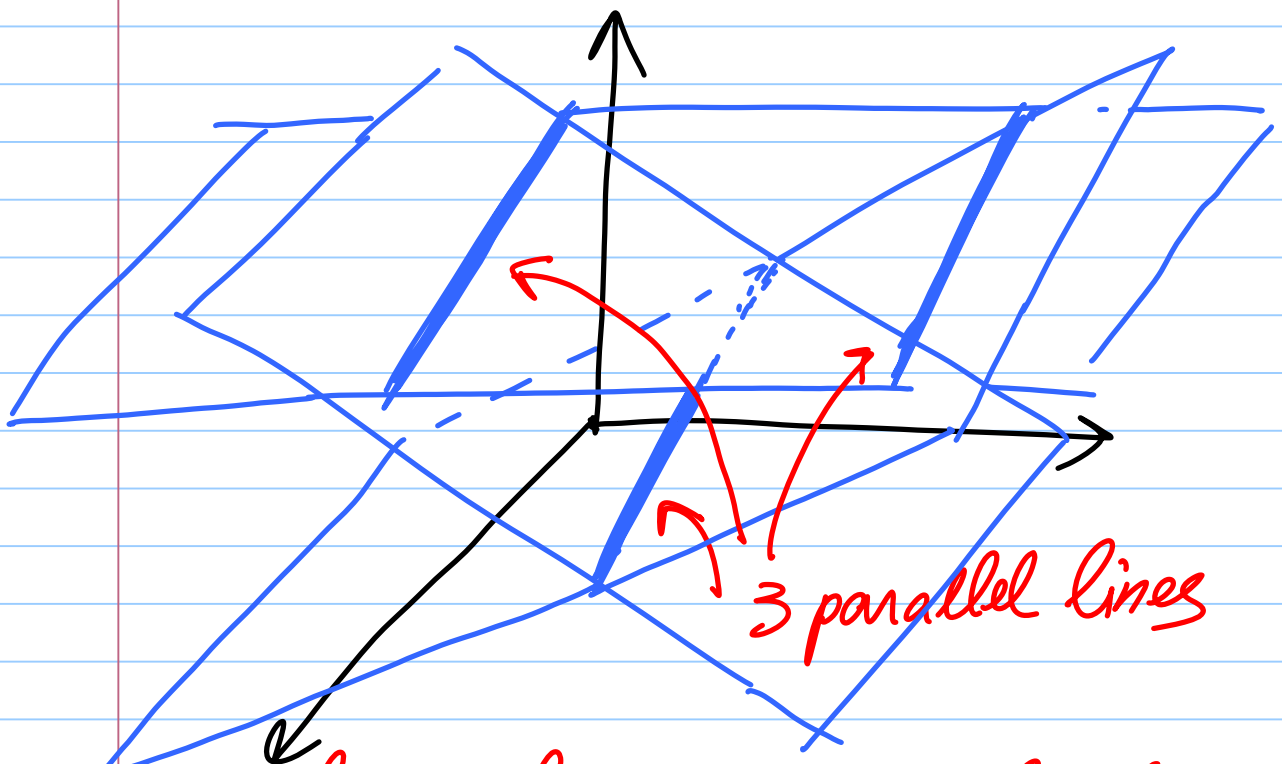
3 planes intersect at a line.

Ex10

$$\begin{cases} x+y-z=2 \\ 2x-y+3z=1 \\ x-2y+4z=0 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 2 & -1 & 3 & 1 \\ 1 & -2 & 4 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & -3 & 5 & -3 \\ 0 & -3 & 5 & -2 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & -3 & 5 & -3 \\ 0 & 0 & 0 & 1 \end{array}\right) \leftarrow \text{no solutions.}$$



3 planes forming 3 parallel lines with no common intersection.