

Column Space (Col(A)) & Null Space (Null(A))

$m \times n$ linear system

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \quad \text{--- (1)}$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \quad \text{--- (2)}$$

$$\begin{matrix} | & | & | & | & | \\ | & | & | & | & | \\ | & | & | & | & | \\ | & | & | & | & | \end{matrix}$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \quad \text{--- (m)}$$

unknowns: $x_1, x_2, \dots, x_n$

Column Space (Col(A)) & Null Space (Null(A))

$m \times n$ linear system

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Column Space (Col(A)) & Null Space (Null(A))

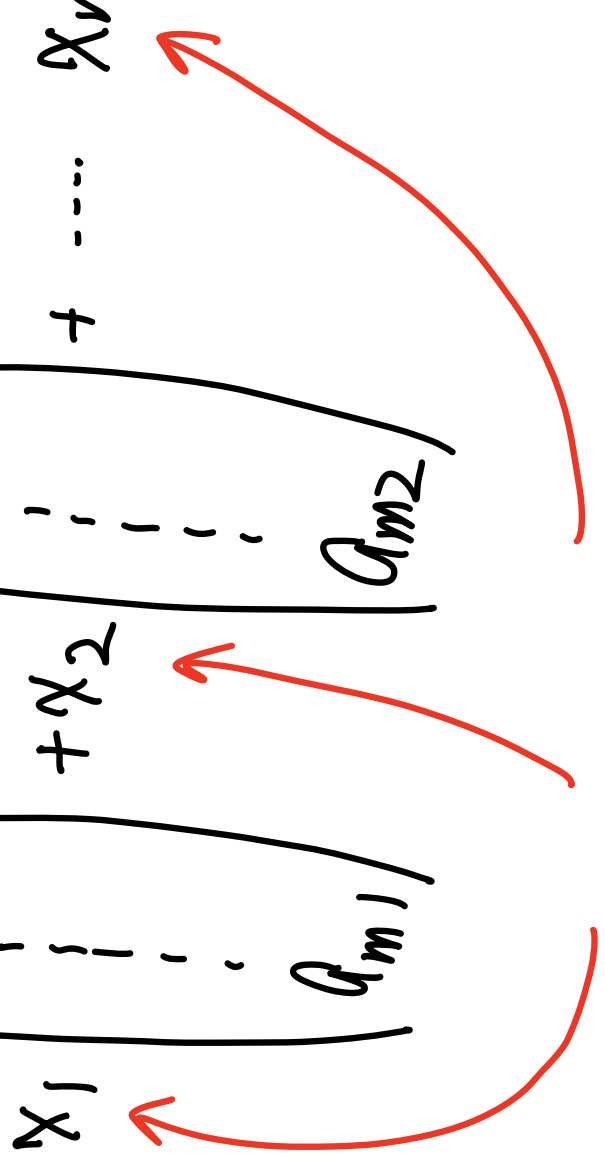
$m \times n$ linear system

$$\left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{array} \right. \quad \begin{array}{l} \text{--- (1)} \\ \text{--- (2)} \\ \text{--- (m)} \end{array}$$

unknowns: $x_1, x_2, \dots, x_n$

Column Space (Col(A)) & Null Space (Null(A))

$m \times n$ linear system

$$\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \dots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$


unknowns: $x_1, x_2, \dots, x_n$

Column Space (Col(A)) & Null Space (Null(A))

$m \times n$ linear system, in matrix form:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$\underbrace{\hspace{15em}}_{A^{(m \times n)}} \quad \underbrace{\hspace{10em}}_{\vec{x} \in \mathbb{R}^n} \quad \underbrace{\hspace{10em}}_{\vec{b} \in \mathbb{R}^m}$

Matrix A multiplied by the vector $\vec{x}$

Column Space (Col(A)) & Null Space (Null(A))

$m \times n$ linear system, in matrix form:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$\underbrace{\hspace{15em}}_{A^{(m \times n)}} \quad \underbrace{\hspace{10em}}_{\vec{x} \in \mathbb{R}^n} \quad \underbrace{\hspace{10em}}_{\vec{b} \in \mathbb{R}^m}$

$\rightarrow a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n$

Column Space (Col(A)) & Null Space (Null(A))

$m \times n$ linear system, in matrix form:

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}}_{A^{(m \times n)}} = \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\vec{x} \in \mathbb{R}^n} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{\vec{b} \in \mathbb{R}^m}$$

$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n$

Column Space (Col(A)) & Null Space (Null(A))

$m \times n$ linear system, in matrix form:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$A^{(m \times n)}$ $\vec{x} \in \mathbb{R}^n$ $\vec{b} \in \mathbb{R}^m$

$\rightarrow a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n$

Column Space (Col(A)) & Null Space (Null(A))

$m \times n$ linear system, in matrix form:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$\underbrace{\hspace{15em}}_{A^{(m \times n)}} \quad \underbrace{\hspace{10em}}_{\vec{x} \in \mathbb{R}^n} \quad \underbrace{\hspace{10em}}_{\vec{b} \in \mathbb{R}^m}$

$\rightarrow a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n$

Properties of the "Notation" $A\vec{x}$

$$A\vec{x} = \vec{b}$$

$m \times n$ ✓

✓ $\in \mathbb{R}^n$

✓ $\in \mathbb{R}^m$

matrix multiplied
by a vector

Linearity Property of Matrix Multiplication

$$(1) \quad A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$$

$$(2) \quad A(\alpha \vec{x}) = \alpha A\vec{x}$$

$$(3) \quad (A+B)(\vec{x}) = A\vec{x} + B\vec{x}$$

$$(4) \quad (\alpha A)(\vec{x}) = \alpha(A\vec{x})$$

Properties of the "Notation" $A\vec{x}$

$$A\vec{x} = \vec{b}$$

$m \times n$ ✓

✓ $\in \mathbb{R}^n$

✓ $\in \mathbb{R}^m$

matrix multiplied
by a vector

Linearity Property of Matrix Multiplication

$$(1) \left\{ \begin{array}{l} A(\alpha\vec{x} + \beta\vec{y}) = \alpha(A\vec{x}) + \beta(A\vec{y}) \\ (2) \end{array} \right.$$

$$(3) \left\{ \begin{array}{l} (\alpha A + \beta B)(\vec{x}) = \alpha(A\vec{x}) + \beta(B\vec{x}) \\ (4) \end{array} \right.$$

Column Space of A and Solution of $AX=b$

$$\left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{array} \right.$$

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

columns of A

Column Space of A and Solution of $A\vec{x} = \vec{b}$

(1)

$$A\vec{x} = \vec{b}$$



$\vec{b}$ can be written as
lin. comb. of columns of A

(2)

Column Space of $A = \text{Span}\{\text{columns of } A\}$
($\text{Col}(A)$)

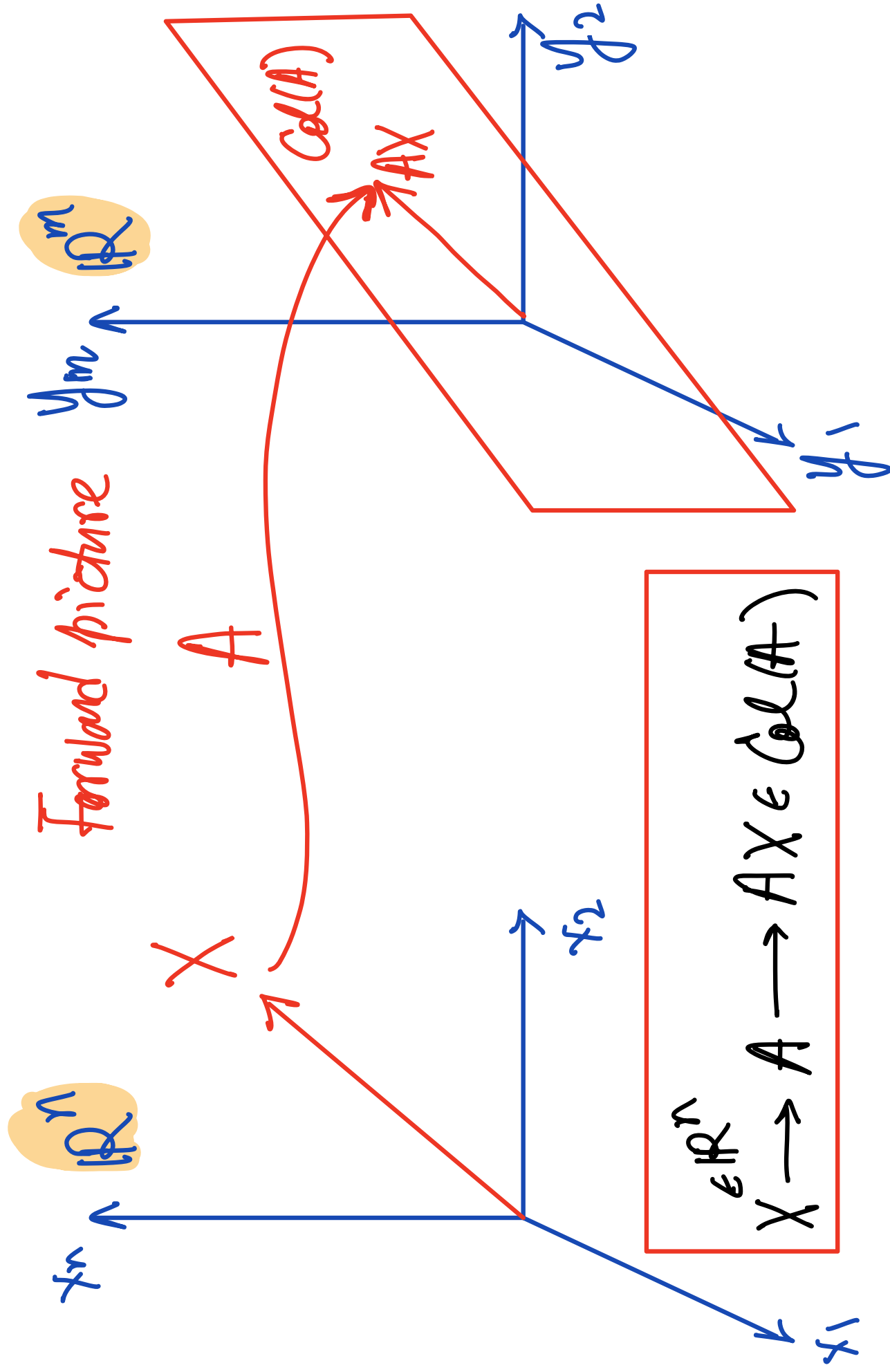
(3)

$$A\vec{x} = \vec{b}$$

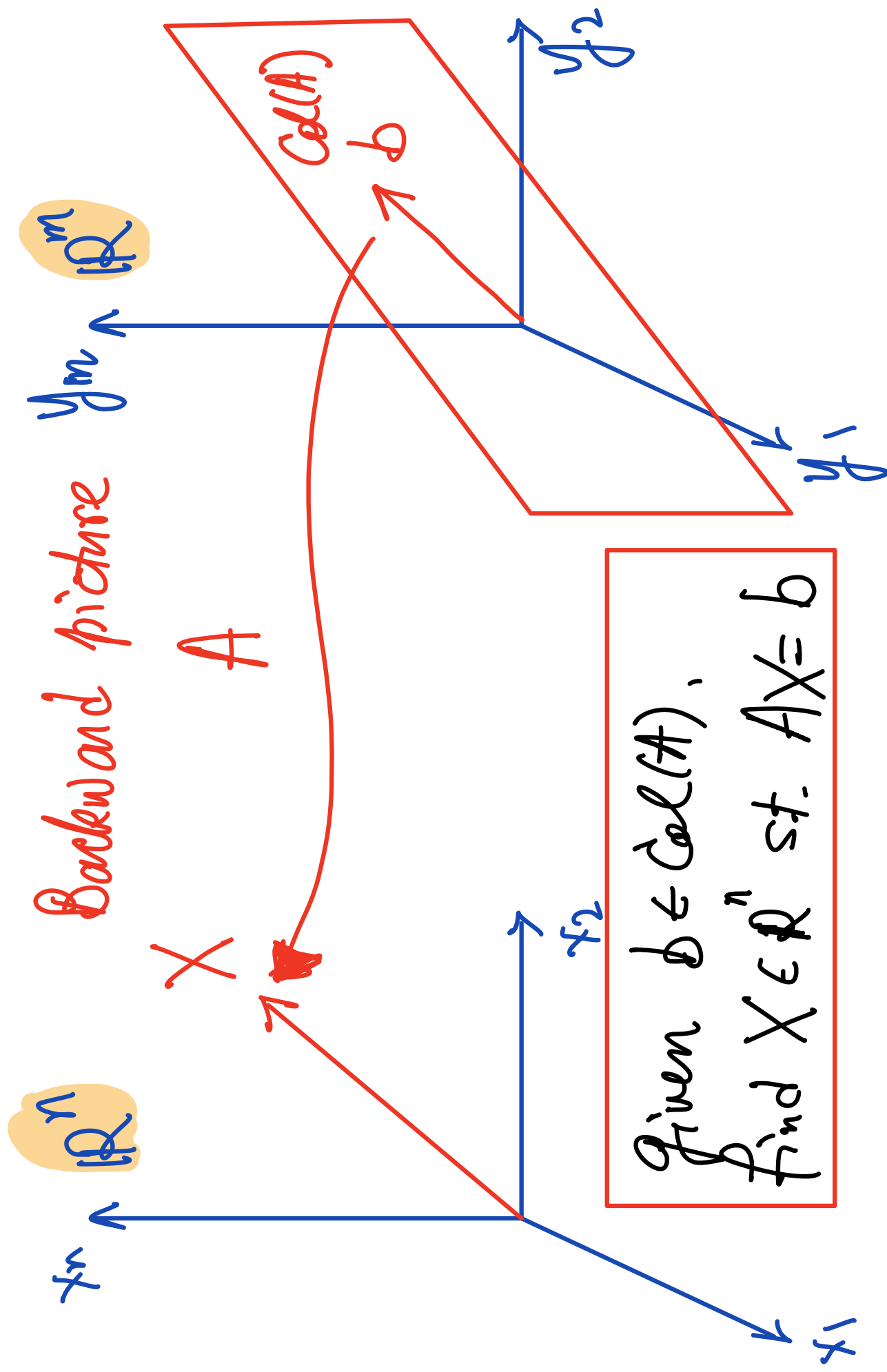


$$\vec{b} \in \text{Col}(A)$$

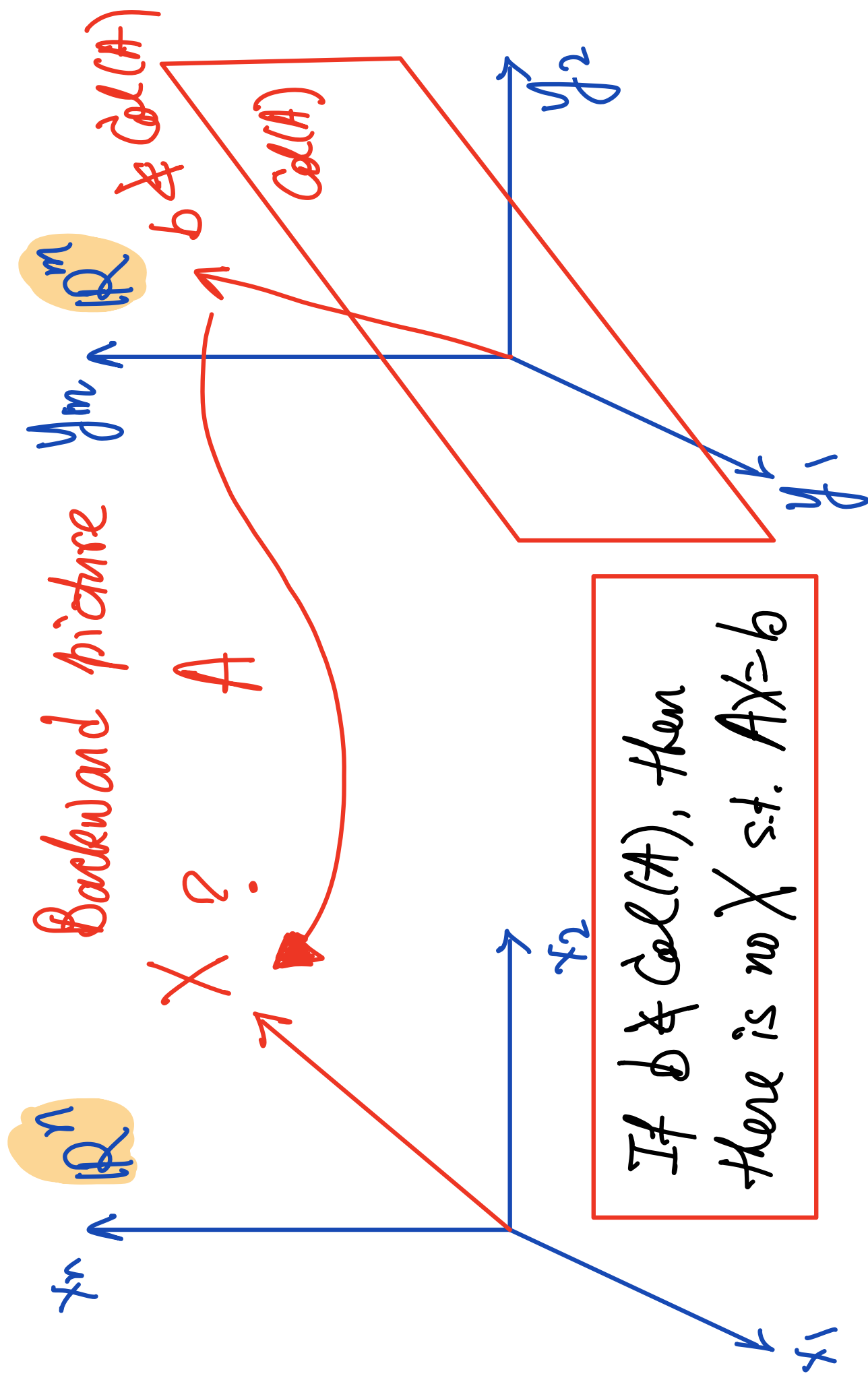
Column Space of A and Solution of $AX=b$



Column Space of A and Solution of $AX=b$



Column Space of A and Solution of $AX=b$



Homogeneous vs Inhomogeneous System

$$A\vec{X} = \vec{0}$$

Homogeneous System (H)

$$\left[A \mid \begin{smallmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{smallmatrix} \right] \rightarrow \rightarrow \rightarrow \left[R \mid \begin{smallmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{smallmatrix} \right]$$

S_i - free variables

$\vec{V}_i$ - spanning vectors

Solution is written as :

$$\vec{X}_h = S_1 \vec{V}_1 + S_2 \vec{V}_2 + \dots + S_k \vec{V}_k$$

$$\text{Null Space of } A (\text{Null}(A)) = \{\text{Solution of } A\vec{X} = \vec{0}\}$$

Homogeneous vs Inhomogeneous System

$$A\vec{x} = \vec{0}$$

Homogeneous System (H)

$$\left[A \mid \begin{smallmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{smallmatrix} \right] \rightarrow \rightarrow \rightarrow \left[R \mid \begin{smallmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{smallmatrix} \right]$$

S_i - free variables

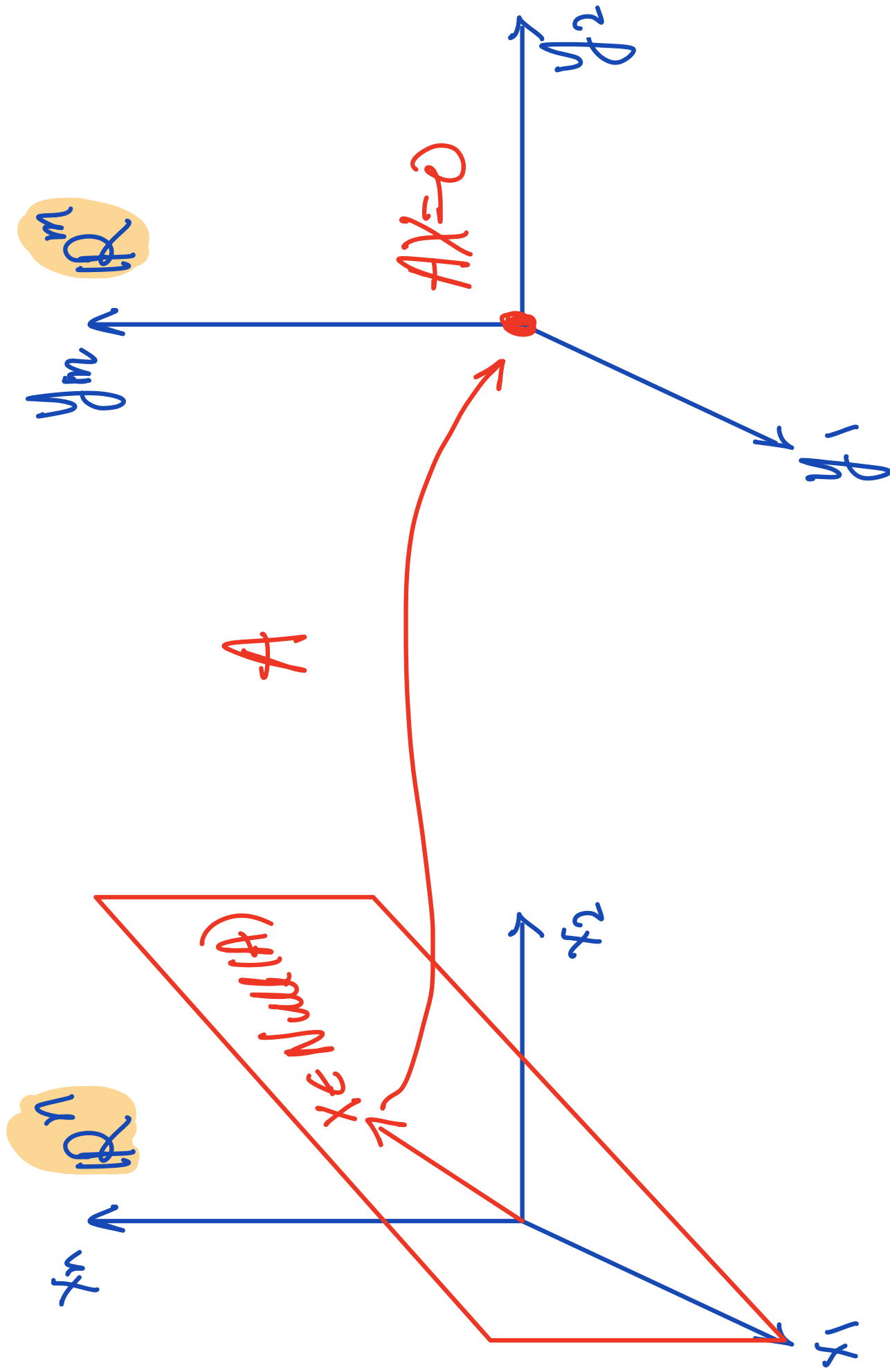
$\vec{v}_i$ - spanning vectors

$$\vec{x}_h = S_1 \vec{v}_1 + S_2 \vec{v}_2 + \dots + S_k \vec{v}_k$$

Solution is written as :

$$\text{Null Space of } A (\text{Null}(A)) = \text{Span of } \vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$$

Homogeneous vs Inhomogeneous System



Homogeneous vs Inhomogeneous System

$$A\vec{x} = \vec{b}$$

Inhomogeneous System (I)

$$[A | \vec{b}] \rightarrow \rightarrow \rightarrow \begin{bmatrix} R \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{bmatrix}$$

Solution is written as: translation vectors spanning vectors

$$\vec{x} = \vec{p} + s_1 \vec{v}_1 + s_2 \vec{v}_2 + \dots + s_k \vec{v}_k$$

Homogeneous vs Inhomogeneous System

$$A\vec{x} = \vec{b}$$

Inhomogeneous System (I)

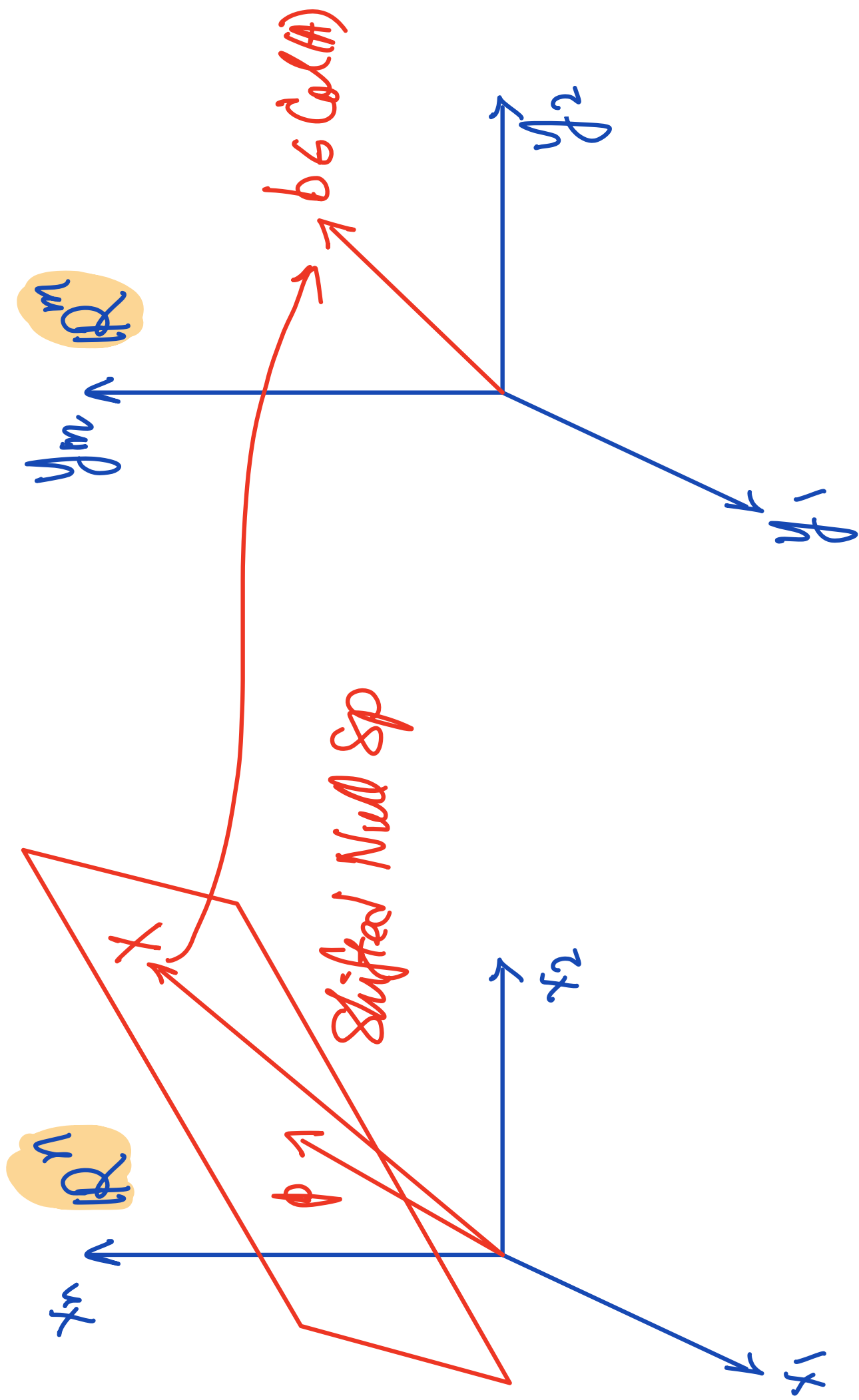
$$[A | \vec{b}] \rightarrow \rightarrow \rightarrow [R | \begin{smallmatrix} * \\ * \\ * \\ * \end{smallmatrix}]$$

shifted Null sp

Solution is written as :

$$\vec{x} = \vec{p} + \text{Null}(A)$$

Homogeneous vs Inhomogeneous System



Solving $m \times n$ Linear System

$$A_{(m \times n)} X_{(n \times 1)} = b_{(m \times 1)}$$

(1) Existence of Solution

$$\iff \vec{b} \in \text{Col}(A)$$

Uniqueness of Solution (if exists)

$$\iff \text{no free variables}$$

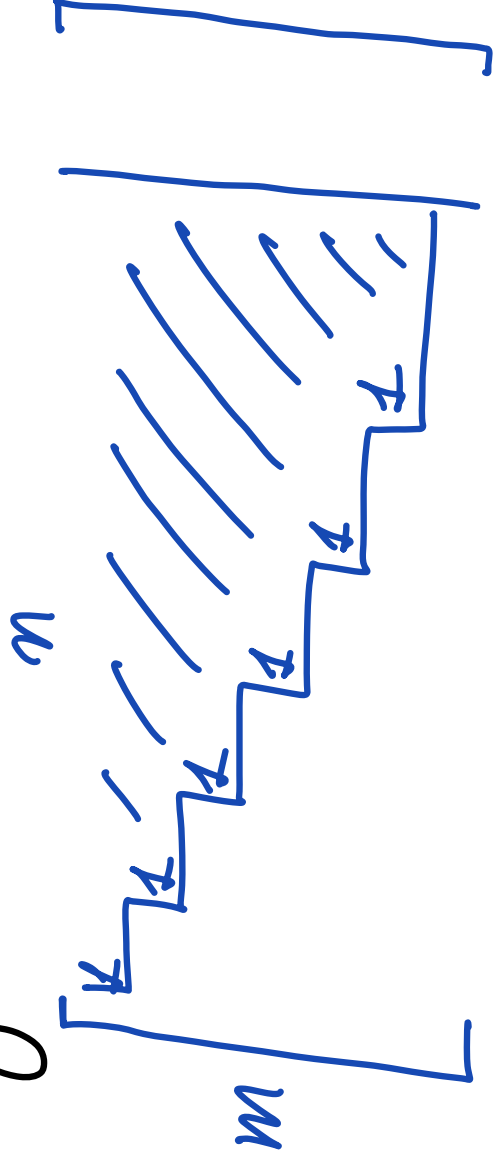
$$\iff \text{Null}(A) = \{\vec{0}\}$$

Solving $m \times n$ Linear System

$$A_{(m \times n)} X_{(n \times 1)} = b_{(m \times 1)}$$

(2) (More unknown Theorem) $n > m$

If solution exists, then it cannot be unique, i.e. there must be some free var.



Solving $m \times n$ Linear System

$$A_{(m \times n)} X_{(n \times 1)} = b_{(m \times 1)}$$

(3) (More equation Theorem) $m > n$

There must be some $\vec{b}$ such that

$AX = \vec{b}$ does not have a solution.

$$\left[\begin{array}{c|c} \text{[shaded rows]} & * \\ \vdots & 0 \\ \hline 0 & 0 \\ \vdots & 0 \end{array} \right]$$