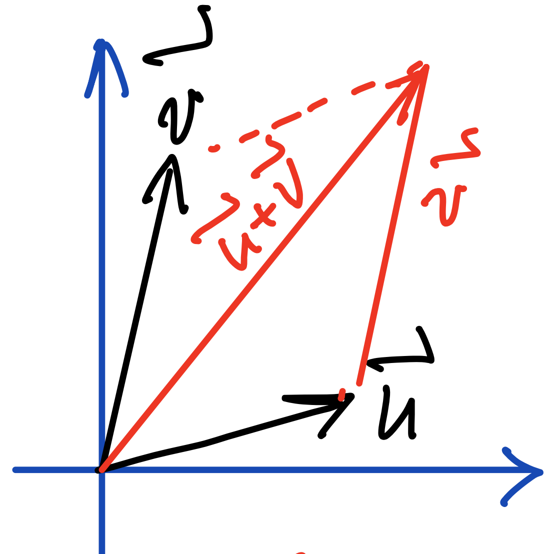
Vectors in $\mathbb{R}^n$

$$\vec{u} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n$$

belong to

Vector Addition

$$\vec{u} + \vec{v} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix}$$



(parallelogram, or "triangular" law)

Vectors in $\mathbb{R}^n$

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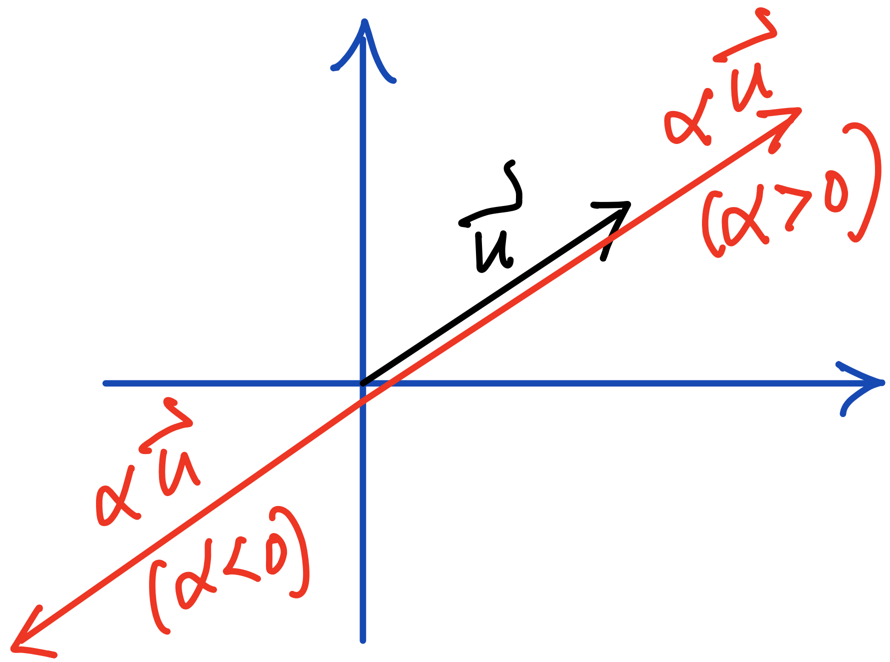
belong to

$\boxed{\text{Scalar Multiplication}}$

$\alpha \in \mathbb{R}$,

$$\alpha \vec{u} = \alpha \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{pmatrix}$$



Properties of Vector Add. & Scalar Mult.

$$\left(\vec{u}+\vec{v} \in \mathbb{R}^n, \quad \alpha\vec{u} \in \mathbb{R}^n \right)$$

1. $\vec{u}+\vec{v} = \vec{v}+\vec{u}$
2. $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$
3. $\alpha(\beta\vec{u}) = (\alpha\beta)\vec{u}$
4. $\alpha(\vec{u}+\vec{v}) = \alpha\vec{u}+\alpha\vec{v}$
5. $(\alpha+\beta)\vec{u} = \alpha\vec{u}+\alpha\vec{v}$

Properties of Vector Add. & Scalar Mult.

$(\vec{u}+\vec{v} \in \mathbb{R}^n, \quad \alpha\vec{u} \in \mathbb{R}^n)$

⑥ Let $\vec{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. Then $\vec{0}+\vec{u}=\vec{u}$

⑦ Let $-\vec{u} = \begin{pmatrix} -x_1 \\ -x_2 \\ \vdots \\ -x_n \end{pmatrix}$. Then $(-\vec{u})+\vec{u}=\vec{0}$

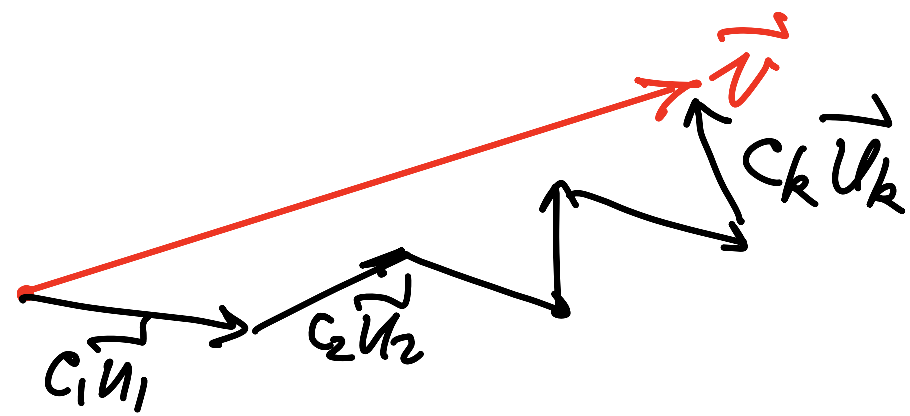
⑧ $1\vec{u} = \vec{u}$

Linear Combination and Span

(i) Given $\mathcal{L} = \{u_1, u_2, \cdots, u_k\} \subseteq \mathbb{R}^n$.

$\vec{v} \in \mathbb{R}^n$ is said to be a $\boxed{\text{linear combination}}$ of $\mathcal{L}$ if there are scalars $c_1, c_2, \cdots, c_k$ such that

$$\boxed{\vec{v} = c_1 \vec{u_1} + c_2 \vec{u_2} + \cdots + c_k \vec{u_k}}$$



Linear Combination and Span

2. Given $\mathcal{L} = \{u_1, u_2, \dots, u_k\} \subseteq \mathbb{R}^n$.

$\boxed{\operatorname{Span}\{u_1, u_2, \dots, u_k\}}$ (or simply $\operatorname{Span}(\mathcal{L})$)

$=$ (the collection of all the possible lin. comb. of the $u_1, u_2, \dots, u_k$)

$$\boxed{= \{ \vec{v} : \vec{v} = c_1 \vec{u_1} + c_2 \vec{u_2} + \dots + c_k \vec{u_k} \}}$$

Linear Combination and Span

Given $\mathcal{L} = \{u_1, u_2, \ldots, u_k\} \subseteq \mathbb{R}^n$.

Main Question:

Given a vector $\vec{v} \in \mathbb{R}^n$, determine if $\vec{v} \in \operatorname{Span}\{\vec{u}_1, \ldots, \vec{u}_k\}$

Find $c_1, c_2, \ldots, c_k$ such that

$$\vec{v} = c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_k\vec{u}_k$$

(General) Vector Space V

V is called a vector space as long as you can define vector addition and scalar mult. such that properties ① — ⑧ hold.

① $\vec{u}+\vec{v}=\vec{v}+\vec{u}$

② $(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$

③ $\alpha(\beta\vec{u})=(\alpha\beta)\vec{u}$

④ $\alpha(\vec{u}+\vec{v})=\alpha\vec{u}+\alpha\vec{v}$

⑤ $(\alpha+\beta)\vec{u}=\alpha\vec{u}+\beta\vec{u}$

⑥ There is $\vec{0}$ such that
$\vec{0}+\vec{u}=\vec{u}$

⑦ For each $\vec{u}$, there is $-\vec{u}$ such that
$\vec{u}+(-\vec{u})=\vec{0}$

⑧ $1\vec{u}=\vec{u}$

(General) Vector Space V

V is called a vector space as long as you can define vector addition and scalar mult. such that properties ① — ⑧ hold.

Important points not to be missed

You need to make sure that

$\boxed{\vec{u}+\vec{v} \in V}$ and $\boxed{\alpha \vec{u} \in V}$

(General) Vector Space V

Examples of (General) Vector Space

① $P_n = \{\text{polynomials of degree at most } n\}$

$$= \{ p(x) \equiv a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \}$$

$$p_1(x) \equiv a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

$$p_2(x) \equiv b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n$$

$$\boxed{(p_1 + p_2)(x) = \underline{(a_0 + b_0)} + \underline{(a_1 + b_1)} x + \cdots + \underline{(a_n + b_n)} x^n}$$

(General) Vector Space V

Examples of (General) Vector Space

① $P_n = \{\text{polynomials of degree at most } n\}$

$$= \{ p(x) \equiv a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \}$$

$$p_1(x) \equiv a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

$$p_2(x) \equiv b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n$$

$$\boxed{(\alpha p_1)(x) \equiv \underline{\alpha a_0} + \underline{\alpha a_1} x + \underline{\alpha a_2} x^2 + \cdots + \underline{\alpha a_n} x^n}$$

(General) Vector Space V

Examples of (General) Vector Space

2. $\mathcal{F} = \{ \text{functions } f : \mathbb{R} \longrightarrow \mathbb{R} \}$

$$x \longrightarrow f(x)$$

$$\boxed{\begin{aligned} (f+g)(x) &= f(x) + g(x) \\ (\alpha f)(x) &= \alpha f(x) \end{aligned}}$$

(General) Vector Space V

Examples of (General) Vector Space

③ $M^{m\times n} = \{\, \underbrace{m}_{\text{no. of rows}} \times \underbrace{n}_{\text{no. of columns}} \text{ matrices} \,\}$

$$= \left\{ A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & & a_{ij} & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (a_{ij}) \right\}$$

(General) Vector Space V

Examples of (General) Vector Space

3 $M^{m\times n} = \{ \underbrace{m}_{\text{no. of rows}} \times \underbrace{n}_{\text{no. of columns}} \text{ matrices} \}$

$$A = (a_{ij}), \quad B = (b_{ij})$$

$$\boxed{\begin{gathered}(A+B) = (a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij}) \\ (\alpha A) = \alpha (a_{ij}) = (\alpha a_{ij})\end{gathered}}$$