Linear Dependence and Independence

Given a list of vectors, $\mathcal{L} = \{u_1, u_2, \dots, u_k\}$

Def 1:

$\mathcal{L}$ is called linearly dependent if one (some) of the vectors can be written as lin. comb. of the rest. (For example, $u_i = \sum_{j \neq i} c_j u_j$)

$\mathcal{L}$ is called linearly independent if none of the vectors can be written as lin. comb. of the rest.

Linear Dependence and Independence

Given a list of vectors, $\mathcal{L} = \{u_1, u_2, ---, u_k\}$

Def 2 : ($\Longleftrightarrow$ Def 1.)

$\mathcal{L}$ is called linearly dependent if there are constants (scalars) $c_1, c_2, ---, c_k$ not all zero (i.e. some non-zero) such that

$$c_1 u_1 + c_2 u_2 + \cdots + c_k u_k = \vec{0}$$

$\mathcal{L}$ is called linearly independent if $c_1 = c_2 = \cdots = c_k = 0$ is the only solution of

$$c_1 u_1 + c_2 u_2 + \cdots + c_k u_k = \vec{0}$$

Linear Dependence and Independence

Given a list of vectors, $\mathcal{L} = \{u_1, u_2, \dots, u_k\}$

$$\boxed{c_1 u_1 + c_2 u_2 + \dots + c_k u_k = \vec{0}}$$

Dependency Relation/Equation

(1) Can determine if u_i's are lin. dep/ind.,

(2) Can find all the relationship between the u_i's

(3) Can determine which vectors to be thrown away.

Consequences (Properties) of Lin Dep.

(I) If $\mathcal{L} = \{u_1, u_2, \dots, u_k\}$ is lin. dep., then $\mathcal{L}$ can be reduced to $\mathcal{L}' (\subseteq \mathcal{L})$ such that $Span(\mathcal{L}) = Span(\mathcal{L}')$.

For example: $u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $u_3 = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$.

$$3u_1 + (-4)u_2 + u_3 = \vec{0}$$

Let $\vec{v} = 5u_1 + 2u_2 + 3u_3$

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$$u_3 = -3u_1 + 4u_2$$

Let
$$\begin{aligned} \vec{v} &= 5u_1 + 2u_2 + 3(-3u_1 + 4u_2) \\ &= (5-9)u_1 + (2+12)u_2 \\ &= -4u_1 + 14u_2 \in \operatorname{Span}(u_1, u_2) \end{aligned}$$

Consequences (Properties) of Lin Dep.

(II) If $\mathcal{L} = \{u_1, u_2, \dots, u_k\}$ is lin. dep., Then any vector $\vec{v} \in \text{Span}(\mathcal{L})$ can be represented in an infinitely many ways as lin. comb. of the u_i's.

i.e. the representation is not unique.

For example,

$$\vec{v} = 2\vec{u_1} - \vec{u_2} + 2\vec{u_3}$$

$+(3\vec{u_1} - 4\vec{u_2} + \vec{u_3} = \vec{0})$

$$= 5\vec{u_1} - 5u_2 + 3\vec{u_3}$$

$-2(3\vec{u_1} - 4\vec{u_2} + \vec{u_3} = \vec{0})$

$$= -\vec{u_1} + 3u_2 + \vec{u_3}$$

Consequences (Properties) of Lin Dep.

(II) $\vec{v} = 2\vec{u}_1 - \vec{u}_2 + 2\vec{u}_3$ $\quad + (3\vec{u}_1 - 4\vec{u}_2 + \vec{u}_3 = \vec{0})$

$= 5\vec{u}_1 - 5u_2 + 3\vec{u}_3$ $\quad -2(3\vec{u}_1 - 4\vec{u}_2 + \vec{u}_3 = \vec{0})$

$= -\vec{u}_1 + 3u_2 + \vec{u}_3$

$$\begin{pmatrix}-4\\7\end{pmatrix} = 2\begin{pmatrix}1\\0\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix} + 2\begin{pmatrix}-3\\4\end{pmatrix}$$

$$= 5\begin{pmatrix}1\\0\end{pmatrix} - 5\begin{pmatrix}0\\1\end{pmatrix} + 3\begin{pmatrix}-3\\4\end{pmatrix}$$

$$= -\begin{pmatrix}1\\0\end{pmatrix} + 3\begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}-3\\4\end{pmatrix}$$

non-unique representation as lin. comb.

How to Eliminate (All the) Redundant Vectors

Given a list $\mathcal{L} = \{u_1, u_2, \dots, u_k\}$.

1. Consider solving for $c_1, c_2, \dots, c_k$:

$$c_1 \vec{u_1} + c_2 \vec{u_2} + \dots + c_k \vec{u_k} = \vec{0}$$

2. Throw away those vectors with the free variable c_i's.

3. The remaining vectors with the pivot variable c_i's are linearly independent.