Basis and Dimension

Given a vector space V, a basis is a list of vectors $\mathcal{B} = \{u_1, u_2, \cdots, u_n\}$ that satisfies:

(1) each $u_i \in V$

(2) $\mathcal{B}$ is lin. ind., i.e.

$$c_1 u_1 + c_2 u_2 + \cdots + c_n u_n = 0 \Rightarrow c_1 = c_2 = \cdots = c_n = 0$$

(3) $\mathcal{B}$ spans/fills the whole V, i.e. for any $\vec{v} \in V$, there are $c_1, c_2, \cdots, c_n$ s.t.

$$c_1 u_1 + c_2 u_2 + \cdots + c_n u_n = v$$

Basis and Dimension

Often (or in fact, always),

(1) Checking for lin. ind. is equivalent to solving a homogeneous system:

$$\boxed{AX = 0}$$

(2) Checking for span is equivalent to solving an inhomogeneous system:

$$\boxed{AX = \vec{b}}$$ for all possible $\vec{b}$

(You do need to find $A, \vec{b}$ for each problem)

Basis and Dimension

The dimension of V, $\dim(V)$ is defined as the number of vectors in a (any) basis.

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$\boxed{\text{Note: There can be many different basis but the dimension is always the same.}}$

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Hence $\dim(\mathbb{R}^2) = 2$

e.g. $\mathbb{R}^2 = \text{Span}\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}\right\}$

(also a basis as it is lin ind. and it spans $\mathbb{R}^2$)

Properties of $\dim(V)$

Let $\dim(V) = n$. Then

Any list of $m > n$ vectors must be linearly dependent, i.e.

n is the maximum number of lin ind. vectors

(Pf: Any linear system with more unknowns than equations must have at least one free var.

More Unknowns Theorem.)

Properties of $\dim(V)$

Let $\dim(V) = n$. Then

Any list of $m > n$ vectors must be linearly dependent, i.e.

n is the maximum number of lin ind. vectors

Any 3 or more vectors in $\mathbb{R}^2$ must be lin dep

Any 4 or more vectors in $\mathbb{R}^3$ must be lin dep

Any 5 or more vectors in $\mathbb{R}^4$ must be lin dep

Properties of $\dim(V)$

Let $\dim(V) = n$. Then

Any list of $m < n$ vectors cannot span V, i.e.

n is the minimum number of vectors that can span V.

Pf: Given $A\vec{X} = \vec{b}$, with # of rows of A > # of cols of A, there is a $\vec{b}$ such that $AX = b$ is not solvable.

More Equations Theorem.

Properties of $\dim(V)$

Let $\dim(V) = n$. Then

$$\boxed{\text{Any list of } m < n \text{ vectors cannot span } V, \text{ i.e. } n \text{ is the minimum number of vectors that can span } V.}$$

Any 2 or fewer vectors cannot span $\mathbb{R}^3$
Any 3 or fewer vectors cannot span $\mathbb{R}^4$
Any 4 or fewer vectors cannot span $\mathbb{R}^5$

Properties of $\dim(V)$

(1) The $\dim(V)$ is the maximum number of vectors that can be lin. ind.

(2) The $\dim(V)$ is the minimum number of vectors that can span the whole V.

(3) Any n vectors that span V must be lin. ind. (Suppose $\dim V = n$)

(4) Any n vectors that are lin ind. must span the whole V. (Suppose $\dim V = n$)

Properties of $\dim(V)$

(1) or (2):

$\boxed{\dim(V) \text{ is a unique number (basis not)}}$

(3) and (4):

In an n-dimensional vector space, any n vectors, lin ind $\Longleftrightarrow$ span $\Leftrightarrow$ basis

The dimension gives the effective degree of freedom in a vector space.

"Standard" Basis

$$\mathbb{R}^2 = \operatorname{Span}\left\{ \underbrace{\begin{pmatrix}1\\0\end{pmatrix}}_{e_1}, \underbrace{\begin{pmatrix}0\\1\end{pmatrix}}_{e_2} \right\}, \qquad \dim(\mathbb{R}^2) = 2$$

$$\begin{pmatrix}x\\y\end{pmatrix} = x\begin{pmatrix}1\\0\end{pmatrix} + y\begin{pmatrix}0\\1\end{pmatrix} = xe_1 + ye_2$$

$$\mathbb{R}^n = \operatorname{Span}\left\{ \underbrace{\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}}_{e_1}, \underbrace{\begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix}}_{e_2}, \cdots, \underbrace{\begin{pmatrix}0\\0\\\vdots\\0\\1\end{pmatrix}}_{e_n} \right\} \qquad \dim(\mathbb{R}^n) = n$$

$$\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix} = x_1\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} + x_2\begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix} + \cdots + x_n\begin{pmatrix}0\\\vdots\\0\\1\end{pmatrix} = x_1e_1 + \cdots + x_ne_n$$

"Standard" Basis

$$M^{2\times 2} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

$$= \left\{ a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$= \mathrm{Span}\left\{ \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{standard basis}} \right\}$$

$$\dim\left(M^{2\times 2}\right) = 4 = (2\times 2)$$

"Standard" Basis

$$M^{m\times n} = \left\{ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} : a_{ij} \in \mathbb{R} \right\}$$

$$= \operatorname{Span}\left\{ \begin{pmatrix} 1 & 0 & \cdots & 0 \\ & 0's & & \end{pmatrix}, \begin{pmatrix} 0 & 1 & \cdots & \\ & 0's & & \end{pmatrix}, \cdots, \cdots \begin{pmatrix} 0 & \cdots & & 0 \\ & 0's & & \\ 0 & \cdots & 0 & 1 \end{pmatrix} \right\}$$

$\dim(M^{m\times n}) = mn$

"Standard" Basis

$$P^n = \{p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n\}$$

$$= \text{Span}\{1, x, x^2, \ldots, x^n\}$$

"Standard" Basis

$$\mathcal{P}^n = \{ p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \}$$

$$= \operatorname{Span}\{1, x, x^2, \ldots, x^n\}$$

($a_0 = a_0 \cdot 1$, 1 is a polynomial of deg. 0.)

$$\boxed{\dim(\mathcal{P}^n) = n+1}$$