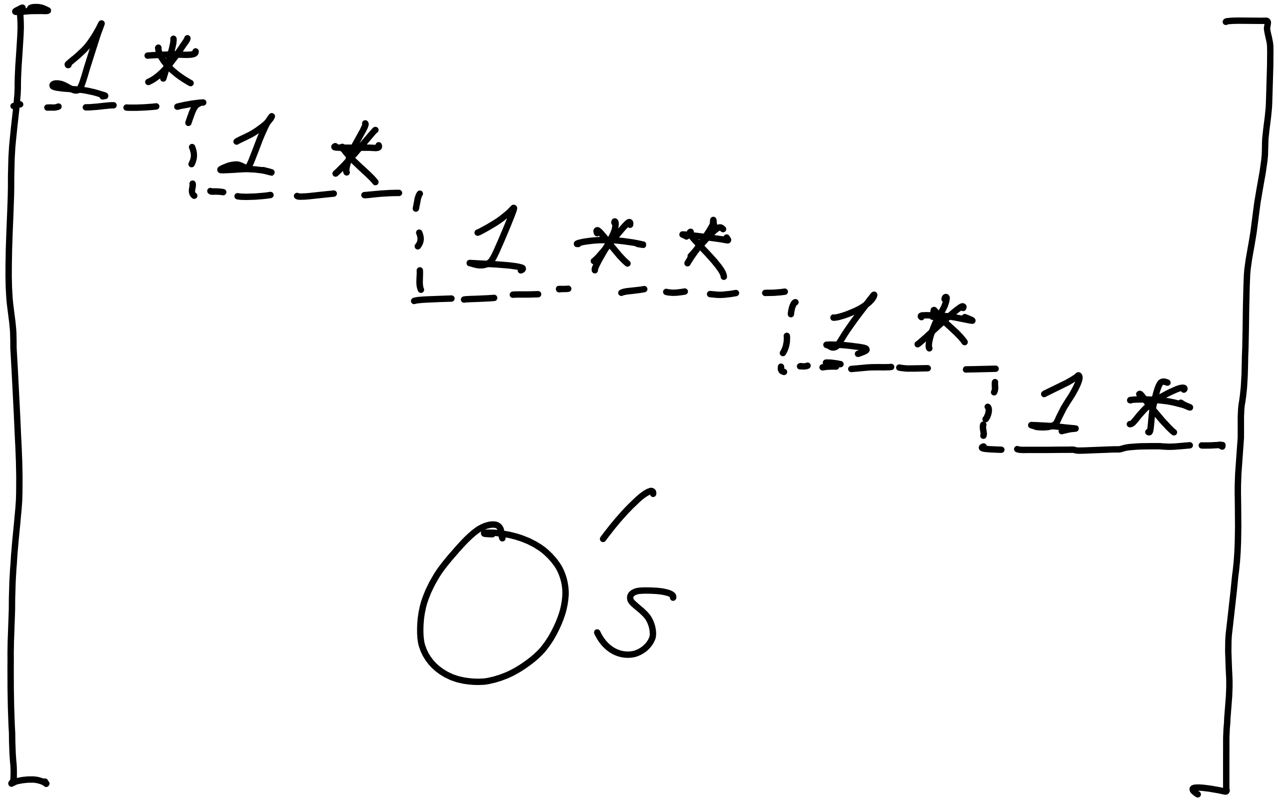
"Anatomy" of Dimensions

$A^{m \times n}$

$$\text{m rows}\left\{ \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{\text{n columns}} \right.$$

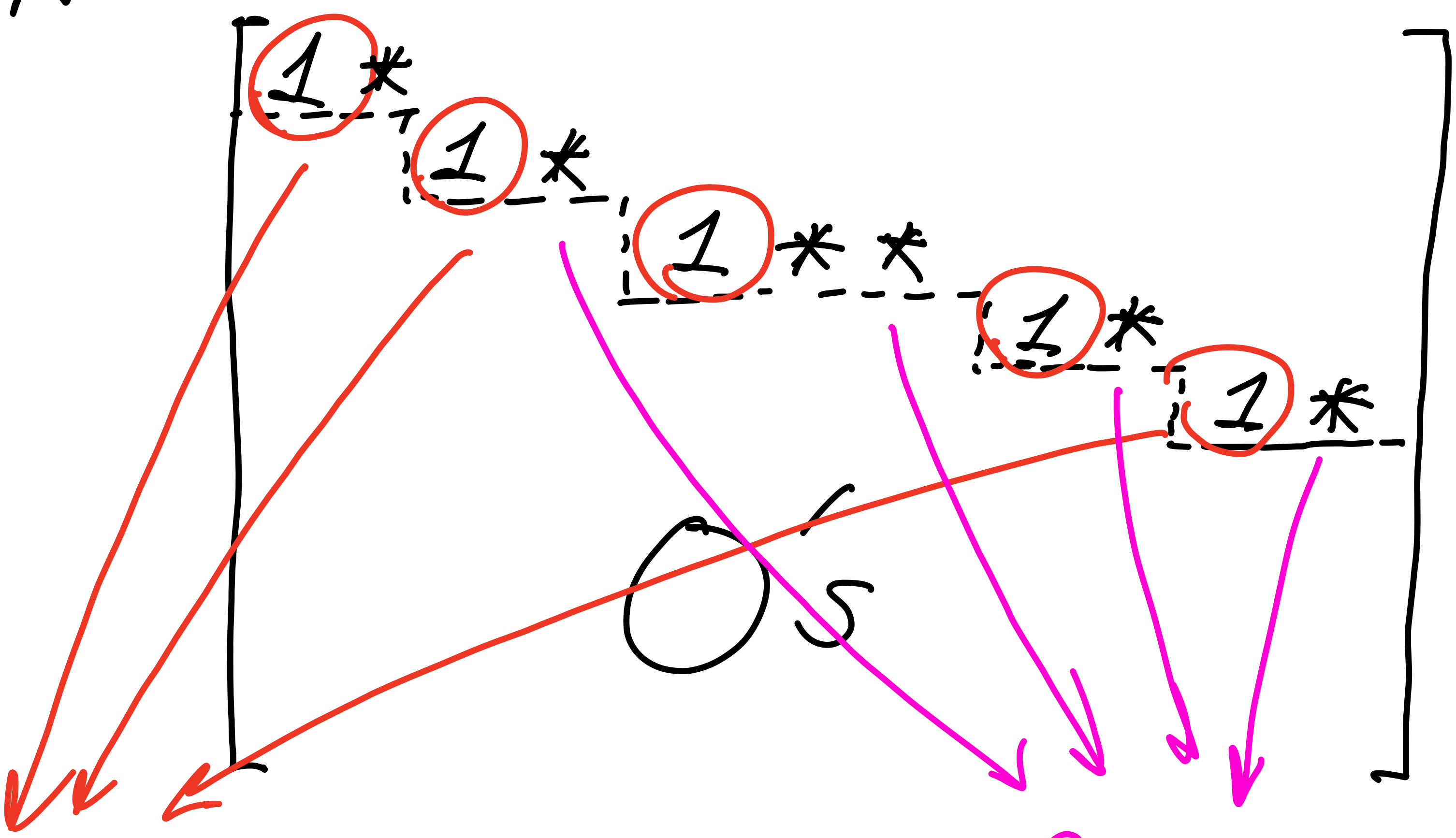
"Anatomy" of Dimensions

$A^{m\times n} \longrightarrow \cdots \longrightarrow R$ ((R)REF of A)



"Anatomy" of Dimensions

$A^{m\times n} \longrightarrow \cdots \longrightarrow R$ ((R)REF of A)



pivot variables

free variables

"Anatomy" of Dimensions

Rank(A) = # of pivot variables

Nullity(A) = # of free variables

Rank-Nullity Theorem:

Rank + Nullity = n (# of columns)

"Anatomy" of Dimensions

(1) $\text{Col}(A)$ = Column space of A

= Span{columns of A} $\subseteq \mathbb{R}^m$

$\dim(\text{Col}(A))$ = # of pivots = Rank

(2) $\text{Null}(A) = \{X : AX = \vec{0}\} \subseteq \mathbb{R}^n$

$\dim(\text{Null}(A))$ = # of free var. = Nullity

"Anatomy" of Dimensions

3 $\text{Row}(A)$ = Row Space of A

$= \text{Span}\{\text{rows of } A\} \subseteq \mathbb{R}^n$

$$\boxed{\text{If } A \xrightarrow[\text{row operations}]{\text{elementary}} \longrightarrow \cdots \longrightarrow B \text{ then } \text{Row}(A) = \text{Row}(B)}$$

$$A \longrightarrow \longrightarrow \longrightarrow \cdots \longrightarrow R_{\text{(R)REF of } A}$$

We have

$$\boxed{\text{Row}(A) = \text{Row}(R)}$$

"Anatomy" of Dimensions

3 Row(A) = Row Space of A

$= \text{Span}\{\text{rows of } A\}$

$= \text{Span}\{\text{rows of } R\} \subseteq \mathbb{R}^n$

$\dim(\text{Row}(A)) = \#$ of non-zero rows of R

$= \#$ of pivots

$= \text{Rank}(A) \quad (= \dim(\text{Col}(A)))$

$$\boxed{\dim(\text{Col}(A)) = \dim(\text{Row}(A))}$$

Relationship to Lin Ind.

Given $A^{m \times n}$,

(1) Columns of A are lin. ind.

$\Longleftrightarrow$ No free variables

$\Longleftrightarrow$ $\text{Rank}(A) = n$

(2) Rows of A are lin. ind

$\Longleftrightarrow$ No zero rows in R

$\Longleftrightarrow$ $\text{Rank}(A) = m$

Relationship to Solving $AX=B$

Given $A^{m\times n}$,

(1) $AX=B$ is solvable for any B

$\iff$ no zeros row in R

$\iff$ $\mathrm{Rank}(A)=m$

(2) $AX=B$ has unique solution (if exists)

$\iff$ no free variables, i.e. only pivots

$\iff$ $\mathrm{Rank}(A)=n$

Non-Singular Matrix

$A^{m \times n}$ is called a non-singular matrix

if for any B, $AX = B$ has a unique solution

$\Longrightarrow$ $\text{Rank}(A) = m$ and $\text{Rank}(A) = n$

$\Longrightarrow$ $m = n$, i.e. A is a square matrix

Non-Singular Matrix

$A^{m\times n}$ is called a non-singular matrix

if for any B, $AX=B$ has a unique solution

$\Longrightarrow$ $\text{Rank}(A) = m$ and $\text{Rank}(A) = n$

$\Longrightarrow$ $m=n$, i.e. A must be a square matrix.

Non-Singular Matrix (Given $A^{n\times n}$)

The following statements are equivalent:

(1) A is non-singular;

(2) $Null(A)=\{0\}$, i.e. no free variables;

(3) $AX=B$ has at most one solution;

(4) $AX=B$ is solvable for any B;

(5) A can be row reduced to I

Non-Singular Matrix (Given $A^{n\times n}$)

The following statements are equivalent:

(6) $\text{Rank}(A) = n$; $(\dim(\text{Col}(A)) = n)$

(7) Columns of A are lin ind;

(8) Columns of A span $\mathbb{R}^n$

(9) $\text{Col}(A) = \mathbb{R}^n$

(10) Columns of A form a basis of $\mathbb{R}^n$

Non-Singular Matrix (Given $A^{n\times n}$)

The following statements are equivalent:

(6) $\text{Rank}(A) = n$; $(\dim(\text{Row}(A)) = n)$

(11) Rows of A are lin ind;

(12) Rows of A span $\mathbb{R}^n$

(13) $\text{Row}(A) = \mathbb{R}^n$

(14) Rows of A form a basis of $\mathbb{R}^n$