Matrix Multiplication

$$A^{m\times n}: \quad X \in \mathbb{R}^n \longrightarrow AX \in \mathbb{R}^m$$

$$AX$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix}$$

$$= x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

Matrix Multiplication

$A^{m\times n}: \quad X \in \mathbb{R}^n \longrightarrow AX \in \mathbb{R}^m$

"Formula" $\boxed{\begin{aligned}(AX)_i &= a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \\ &= \sum_{k=1}^{n} a_{ik}x_k\end{aligned}}$

Linearity Properties

(1) $A(\underline{\alpha X + \beta Y}) = \alpha AX + \beta AY$

(2) $(\underline{\alpha A + \beta B})X = \alpha AX + \beta BX$

Composition of Linear Transformations

$$\begin{array}{ccccc} \mathbb{R}^l & \xrightarrow{B^{m\times l}} & \mathbb{R}^m & \xrightarrow{A^{n\times m}} & \mathbb{R}^n \\ X & \longrightarrow & Y = BX & \longrightarrow & Z = AY \end{array}$$

$$= A(BX)$$

$$= \underline{(AB)}X$$

$$Z = AY = A(BX)$$

$$= A(x_1 B_1 + x_2 B_2 + \cdots + x_l B_l) \quad (B_j - j^{th} \text{ col of } B)$$

$$= x_1 AB_1 + x_2 AB_2 + \cdots + x_l AB_l$$

$$= \underline{\begin{bmatrix} AB_1 & AB_2 & \cdots & AB_l \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_l \end{bmatrix}$$

Composition of Linear Transformations

$$\mathbb{R}^{\ell} \xrightarrow{B^{m\times \ell}} \mathbb{R}^{m} \xrightarrow{A^{n\times m}} \mathbb{R}^{n}$$

$$X \longrightarrow Y = BX \longrightarrow Z = AY$$

$$= A(BX)$$

$$= (AB)X$$

$$\boxed{AB := [AB_1 \; AB_2 \; \cdots \; AB_{\ell}]}$$

Matrix Multiplication

$$A^{n\times m} = (A_{ij}), \quad B^{m\times l} = (B_{ij})$$

$$A^{n\times m} B^{m\times l} = C^{n\times l}$$

i^{th} row of A

j^{th} col. of B

$$C_{ij} = (AB)_{ij} = \begin{pmatrix} A_{i1} & A_{i2} & \cdots & A_{im} \\ & & & \end{pmatrix} \begin{pmatrix} & B_{1j} & \\ & B_{2j} & \\ & \vdots & \\ & B_{mj} & \end{pmatrix}$$

$$= A_{i1}B_{1j} + A_{i2}B_{2j} + \cdots + A_{im}B_{mj}$$

$$= \sum_{k=1}^{m} A_{ik}B_{kj}$$

(Same as $\circledast$)

Properties of Matrix Multiplication

(1) $A(\alpha B + \beta C) = \alpha AB + \beta AC$

$(\alpha A + \beta B)C = \alpha AC + \beta BC$

(2) $A(BC) = (AB)C$

(3) $(AB)^T = B^T A^T$

(4) In general, $AB \neq BA$

Properties of Matrix Multiplication

(5) Identity Matrix (Square)

$$I_n = \begin{pmatrix} 1 & & & \\ & 1 & & 0's \\ & 0's & \ddots & \\ & & & 1 \end{pmatrix}^{n\times n}$$

$$I_1 = (1), \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$I_n A^{n\times m} = A^{n\times m}, \qquad A^{n\times m} I_m = A^{n\times m}$$

Properties of Matrix Multiplication

(6) Diagonal Matrix (square)

$$A^{n\times n} = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix}, \quad B^{n\times n} = \begin{pmatrix} b_1 & & & 0 \\ & b_2 & & \\ & & \ddots & \\ 0 & & & b_n \end{pmatrix}$$

$$AB = \begin{pmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{pmatrix} \begin{pmatrix} b_1 & & & 0 \\ & b_2 & & \\ & & \ddots & \\ 0 & & & b_n \end{pmatrix} = \begin{pmatrix} a_1b_1 & & & 0 \\ & a_2b_2 & & \\ & & \ddots & \\ 0 & & & a_nb_n \end{pmatrix}$$