Eigenvalues and Eigenvectors

Given $A^{n \times n}$, $X (\neq 0) \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ is called and eigenvector and eigenvalue pair if

$$\boxed{AX = \lambda X}$$

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Given $A^{n\times n}$, $X(\neq 0) \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ is called and eigenvector and eigenvalue pair if

$$\boxed{AX = \lambda X} \qquad (X \neq 0)$$

$$
\begin{aligned}
(A-\lambda I)X = 0 &\iff X \in Null(A-\lambda I) \\
&\iff Null(A-\lambda I) \text{ is non-trivial} \\
&\iff (A-\lambda I)^{-1} \text{ does not exist} \\
&\iff (A-\lambda I) \text{ is singular}
\end{aligned}
$$

Finding Eigenvalues and Eigenvectors

$(A-\lambda I)$ is singular (i.e. $(A-\lambda I)^{-1}$ does not exist)

$\iff \det(A-\lambda I)=0$

$p(\lambda)=\det(A-\lambda I)$: characteristic polynomial

(1) $p(\lambda)=c(\lambda-\lambda_1)^{m_1}(\lambda-\lambda_2)^{m_2}\cdots(\lambda-\lambda_k)^{m_k}=0$

(2) λ_i are the eigenvalues (repeating m_i times)

(3) For each λ_i, solve: $(A-\lambda_i I)X=\vec{0}$.

Eigenspace, Algebraic and Geometric Multiplicities

(1) **Eigenspace** w.r.t. $\lambda_i := Null(A - \lambda_i I)$

(All the eigenvectors w.r.t. λ_i form a <u>subspace</u>)

(2) **Geometric Multiplicity** w.r.t. λ_i (g_i)

$g_i = \dim(Null(A - \lambda_i I))$

$=$ # of lin. ind. eigenvectors w.r.t. λ_i

(3) **Algebraic Multiplicity** of λ_i (m_i)

$m_i =$ # of times λ_i repeats in $p(\lambda)$

Eigenspace, Algebraic and Geometric Multiplicities

For each λ_i, $\boxed{1 \leq g_i \leq m_i}$

(1) If $\boxed{g_i < m_i}$, then λ_i is called defective;
(deficient)

If $\boxed{g_i = m_i}$, then λ_i is called non-defective
(non-deficient)

(2) If $\boxed{g_i = m_i \text{ for all } i}$, then
A is called non-defective (diagonalizable)

Properties of $\{(x_i, \lambda_i)\}$

(1) Eigenvalues of (upper/lower-) triangular matrices = diagonal entries

$$A = \begin{pmatrix} a_{11} & & a_{ij} \\ & a_{22} & \\ 0\text{'s} & & \ddots \quad a_{nn} \end{pmatrix}, \qquad A - \lambda I = \begin{pmatrix} a_{11}-\lambda & & \\ & a_{22}-\lambda & a_{ij} \\ 0\text{'s} & & \ddots \\ & & & a_{nn}-\lambda \end{pmatrix}$$

$$\boxed{\det(A-\lambda I) = (a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda)}$$

Properties of $\{(x_i, \lambda_i)\}$

(2) $\quad tr(A) := \sum_i a_{ii} = \sum_{i=1}^{n} \lambda_i$ (counting mult.)

$\det(A) = \prod_{i=1}^{n} \lambda_i$ (counting multiplicities)

$$\det(A - \lambda I) := p(\lambda) = (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$$

$$= (-1)^n \left[\lambda^n - \underbrace{(\lambda_1 + \lambda_2 + \cdots + \lambda_n)}_{= tr(A)} \lambda^{n-1} + \cdots + (-1)^n \underbrace{\lambda_1 \lambda_2 \cdots \lambda_n}_{= \det(A)} \right].$$

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$\det(A) = \prod_{i=1}^{n} \lambda_i$ (counting multiplicities)

$$\det(A-\lambda I) = \det\begin{pmatrix} a_{11}-\lambda & & \\ & a_{22}-\lambda & a_{ij} \\ a_{ij} & & \ddots \\ & & & a_{nn}-\lambda \end{pmatrix} = \cdots\cdots$$

$$= (-1)^n\left[\lambda^n - (a_{11}+a_{22}+\cdots+a_{nn})\lambda^{n-1} + \cdots\cdots + \det A\right]$$

Properties of $\{(x_i, \lambda_i)\}$

(3) Eigenvectors w.r.t. distinct λ_i are lin ind.

$$\det(A-\lambda I) = (-1)^n (\lambda-\lambda_1)^{m_1} (\lambda-\lambda_2)^{m_2} \cdots (\lambda-\lambda_k)^{m_k}$$

$$x_1 \qquad x_2 \qquad x_k$$

Then $\{x_1, x_2, \cdots, x_k\}$ are lin. ind.

Properties of $\{(X_i, \lambda_i)\}$

(3) Eigenvectors w.r.t. distinct λ_i are lin ind.

$$\det(A-\lambda I) = (-1)^n(\lambda-\lambda_1)^{m_1}(\lambda-\lambda_2)^{m_2}\cdots(\lambda-\lambda_k)^{m_k}$$

$$\underbrace{\{\underbrace{X_1^{(1)}, X_1^{(2)}, \cdots X_1^{(g_1)}}_{g_1}\}, \{\underbrace{X_2^{(1)}, X_2^{(2)}, \cdots, X_2^{(g_2)}}_{g_2}\}, \cdots \{\underbrace{X_k^{(1)}, \cdots, X_k^{(g_k)}}_{g_k}\}}_{\text{Total} = g_1+g_2+\cdots+g_k \quad \text{(still) are lin ind.}}$$