Diagonalization of a matrix

Given $A^{n\times n}$ (assume diagonalizable, $g_i = m_i$ for all i)

$$AX_i = \lambda_i X_i, \qquad i = 1, 2, \dots, n$$

$$\begin{bmatrix} AX_1 & AX_2 & \cdots & AX_n \end{bmatrix} = \begin{bmatrix} \lambda_1 X_1 & \lambda_2 X_2 & \cdots & \lambda_n X_n \end{bmatrix}$$

$$A \underbrace{\begin{bmatrix} X_1 & \cdots & X_n \end{bmatrix}}_{Q} = \underbrace{\begin{bmatrix} X_1 & \cdots & X_n \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}}_{D}$$

$$\boxed{A = QDQ^{-1}}$$

Ex 1 $\quad A = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$,

$$\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \end{bmatrix} = 2\begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}^{-1}}$$

Ex 2 $\quad A = \begin{bmatrix} 5 & -2 \\ 3 & 0 \end{bmatrix}$,

$$\begin{bmatrix} 5 & -2 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -2 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} 5 & -2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 2 & \\ & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}^{-1}}$$

Ex3

$$A = \begin{bmatrix} 15 & -10 \\ 15 & -10 \end{bmatrix},$$

$$\begin{bmatrix} 15 & -10 \\ 15 & -10 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 0\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 15 & -10 \\ 15 & -10 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} 15 & -10 \\ 15 & -10 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 0 & \\ & 5 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}^{-1}}$$

Ex 4

$$A = \begin{bmatrix} 0 & 3 & -3 \\ 2 & 2 & -2 \\ -4 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 3 & -3 \\ 2 & 2 & -2 \\ -4 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 0\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 3 & -3 \\ 2 & 2 & -2 \\ -4 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = -3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 3 & -3 \\ 2 & 2 & -2 \\ -4 & -1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = 6\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} 0 & 3 & -3 \\ 2 & 2 & 2 \\ -4 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & & \\ & -3 & \\ & & -6 \end{bmatrix}\begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}^{-1}}$$

Ex5 $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$

$$\begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = 1\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & & \\ & 1 & \\ & & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}^{-1}}$$

Interpretation of Diagonalization : $A = QDQ^{-1}$

(Change of Basis)

Let $b \in \mathbb{R}^n$.

Write $\underline{b = c_1 X_1 + c_2 X_2 + \cdots + c_n X_n}$

Then $Ab = A(c_1 X_1 + c_2 X_2 + \cdots + c_n X_n)$

$$= c_1 A X_1 + c_2 A X_2 + \cdots + c_n A X_n$$

$$= \underline{c_1 \lambda_1 X_1 + c_2 \lambda_2 X_2 + \cdots + c_n \lambda_n X_n}$$

① $b = c_1 X_1 + \cdots + c_n X_n = \underbrace{\begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}}_{C}$

Coordinates of b in the basis $\{X_1 X_2 \ldots X_n\}$

$b = QC \implies C = Q^{-1} b$

②

$$Ab = c_1\lambda_1 X_1 + c_2\lambda_2 X_2 + \cdots + c_n\lambda_n X_n$$

coordinates of Ab in the basis $\{X_1, X_2, \ldots, X_n\}$

$$= \begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix} \underbrace{\begin{bmatrix} c_1\lambda_1 \\ c_2\lambda_2 \\ \vdots \\ c_n\lambda_n \end{bmatrix}}$$

$$= \underbrace{\begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix}}_{Q} \overbrace{\underbrace{\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}}_{C}}$$

$$= \underbrace{Q \underbrace{D \underbrace{Q^{-1} b}_{\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}}}_{\begin{bmatrix} \lambda_1 c_1 \\ \lambda_2 c_2 \\ \vdots \\ \lambda_n c_n \end{bmatrix}}}_{\lambda_1 c_1 X_1 + \lambda_2 c_2 X_2 + \cdots + \lambda_n c_n X_n}$$

Application of Diagonalization

Computation of A^k

$$A = QDQ^{-1}$$

$$A^2 = QD\cancel{Q^{-1}}\cancel{Q}DQ^{-1} = QD^2Q^{-1}$$

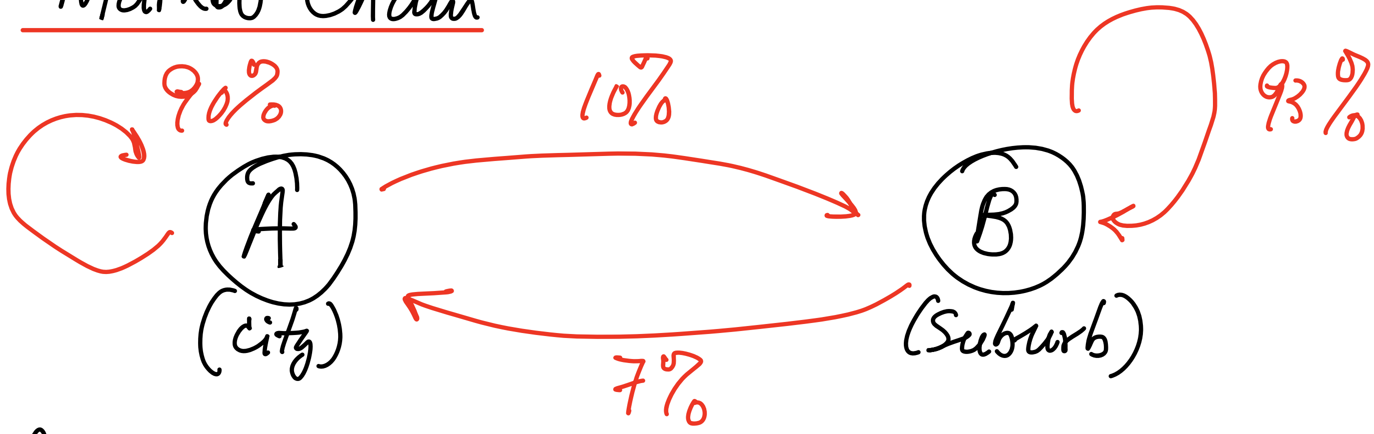
$$A^3 = QD\cancel{Q^{-1}}\cancel{Q}D\cancel{Q^{-1}}\cancel{Q}DQ^{-1} = QD^3Q^{-1}$$

$$\vdots \quad \vdots \quad \vdots$$

$$\boxed{A^k = QD^kQ^{-1}}$$

$$D^k = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}^k = \begin{bmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k \end{bmatrix}$$

Markov Chain



Let a_n = population in A in year n

b_n = population in B in year n

Let $a_0 = 38\,(K)$, $b_n = 62\,(K)$

$$a_1 = 0.9a_0 + 0.07b_0$$
$$b_1 = 0.1a_0 + 0.93b_0$$

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0.9 & 0.07 \\ 0.1 & 0.93 \end{bmatrix} \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}$$

$$a_2 = 0.9a_1 + 0.07b_1$$
$$b_2 = 0.1a_1 + 0.93b_1$$

$$\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0.9 & 0.07 \\ 0.1 & 0.93 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.9 & 0.07 \\ 0.1 & 0.93 \end{bmatrix}^2 \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}$$

Let $P_0 = \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}$, $P_1 = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$, ... $P_n = \begin{bmatrix} a_n \\ b_n \end{bmatrix}$

Then
$$P_1 = AP_0$$
$$P_2 = AP_1 = A^2 P_0$$
$$P_3 = AP_2 = A^3 P_0$$
$$\vdots \quad \vdots$$
$$\boxed{P_n = A^n P_0}$$

$$\boxed{Q: \lim_{n\to\infty} P_n = \lim_{n\to\infty} A^n P_0 = ?}$$

<u>Ans:</u> <u>Find λ, X for A</u>

$$\det(A - \lambda I) = \det\begin{bmatrix} 0.9-\lambda & 0.07 \\ 0.1 & 0.93-\lambda \end{bmatrix}$$
$$= (0.9-\lambda)(0.93-\lambda) - 0.007$$
$$= \lambda^2 - 1.83\lambda + 0.83$$
$$= (\lambda - 1)(\lambda - 0.83)$$

$\lambda = 1,\ 0.83$

$\underline{\lambda_1 = 1}$:

$$(A - I)X_1 = 0 \Rightarrow \left[\begin{array}{cc|c} -0.1 & 0.07 & 0 \\ 0.1 & -0.07 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & -0.7 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$X_1 = \begin{bmatrix} 0.7\alpha \\ \alpha \end{bmatrix} \overset{\alpha=1}{=} \begin{bmatrix} 0.7 \\ 1 \end{bmatrix}$$

$\lambda_2 = 0.83$

$$(A - 0.83I)X_2 = 0 \Rightarrow \left[\begin{array}{cc|c} 0.07 & 0.07 & 0 \\ 0.1 & 0.1 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Hence

$$\boxed{\begin{aligned} A &= \begin{bmatrix} 0.7 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.83 \end{bmatrix} \begin{bmatrix} 0.7 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \\ A^n &= \begin{bmatrix} 0.7 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1^n & \\ & 0.83^n \end{bmatrix} \begin{bmatrix} 0.7 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \\ \lim_{n\to\infty} A^n &= \begin{bmatrix} 0.7 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0.7 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \end{aligned}}$$

$$P_n = A^n P_0 = \begin{bmatrix} 0.7 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & 0.83^n \end{bmatrix} \begin{bmatrix} 0.7 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 38 \\ 62 \end{bmatrix}$$

$$= \begin{bmatrix} 0.7 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & 0.83^n \end{bmatrix} \frac{\begin{bmatrix} 1 & 1 \\ -1 & 0.7 \end{bmatrix}}{1.7} \begin{bmatrix} 38 \\ 62 \end{bmatrix}$$

$$= \begin{bmatrix} 0.7 & -0.83^n \\ 1 & 0.83^n \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0.7 \end{bmatrix} \begin{bmatrix} 38 \\ 62 \end{bmatrix} \frac{1}{1.7}$$

$$= \begin{bmatrix} 0.7 + 0.83^n & 0.7 - 0.7(0.83)^n \\ 1 - 0.83^n & 1 + 0.7(0.83)^n \end{bmatrix} \begin{bmatrix} 38 \\ 62 \end{bmatrix} \frac{1}{1.7}$$

$$= \frac{1}{1.7} \begin{bmatrix} 38(0.7 + 0.83^n) + 62(0.7 - 0.7(0.83)^n \\ 38(1 - 0.83^n) + 62(1 + 0.7(0.83)^n \end{bmatrix}$$

$$= \frac{1}{1.7} \begin{bmatrix} 70 - 5.4(0.83)^n \\ 100 + 5.4(0.83)^n \end{bmatrix}$$

$$\lim_{n\to\infty} P_n = \lim_{n\to\infty} A^n P_0 = \frac{1}{1.7} \begin{bmatrix} 70 \\ 100 \end{bmatrix}$$ ($\Leftarrow$ steady state)