Beginning .....

Given $A^{m\times n}$, $\vec{b}^{\,n\times 1}$, solve for $\vec{X}^{\,n\times 1}$:

$$\boxed{A\vec{X} = \vec{b}}$$

There are three possibilities:

(1) unique solution

(2) infinitely many solutions

(3) no solution

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- use row reduction
- row echelon form

Rank(A) & Nullity(A)

Rank(A) ⟶ no. of pivot variables

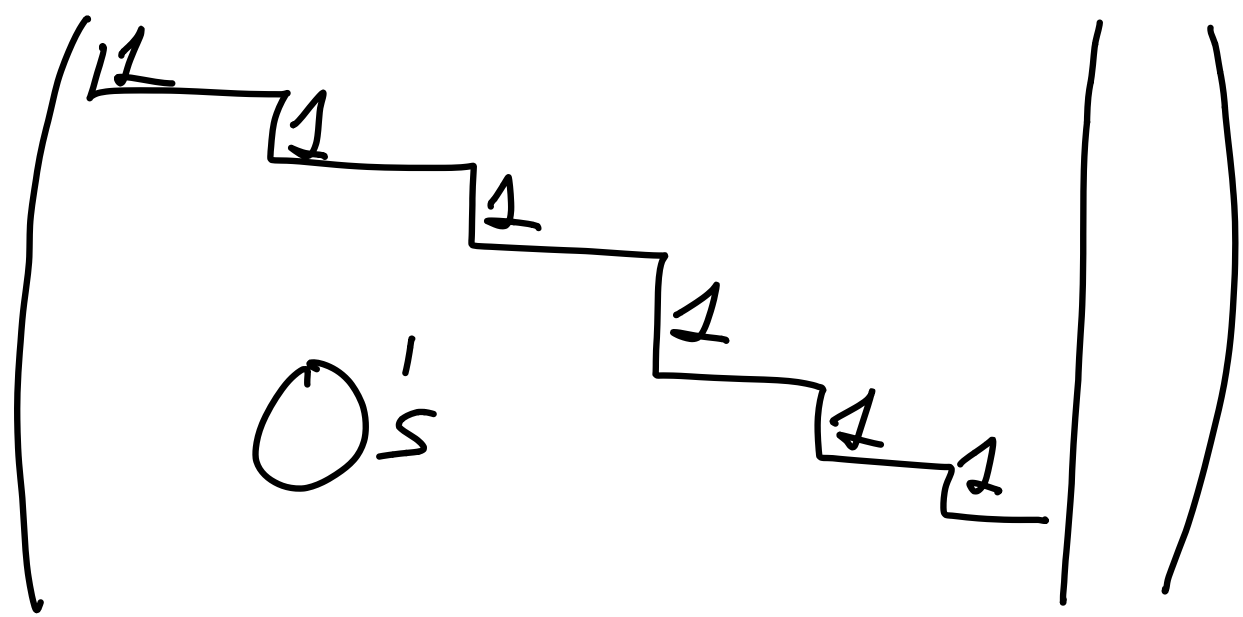
Nullity(A) ⟶ no. of free variables

$$\text{Rank}(A) + \text{Nullity}(A) = n$$

$(A|b) \longrightarrow$ row operations

$\longrightarrow$ (reduced) row echelon form

$\longrightarrow$



Existence and Uniqueness of Solution

(1) $\boxed{A^{m\times n} x = b \text{ is solvable for any } \vec{b}}$

$\Updownarrow$

# of pivots $= m$; $\text{Rank}(A) = m$

(2) $A^{m\times n} x = b$ has at most one solution
(has unique solution, if exists)

$\Updownarrow$

# of free variables $= 0$; $\text{Rank}(A) = n$

Linear Combination

(1) Given $\{u_1, u_2, \dots, u_n\}$,

$$\boxed{\vec{v} \in \mathrm{Span}\{u_1, \dots, u_n\}}$$

$$\Updownarrow$$

$$\boxed{\vec{v} = c_1\vec{u_1} + c_2\vec{u_2} + \dots + c_n\vec{u_n}}$$

$c_1, c_2, \dots, c_n$ can be found.

Linear Combination

(2) $\boxed{\text{Solving } AX = b}$

$\Updownarrow$

$\boxed{\text{express } \vec{b} \text{ as lin comb. of columns of } A.}$

$$AX = b \iff x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{21} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Linear Independence & Dependence

Given $\{u_1, u_2, \cdots, u_n\}$

1. Linear independence

   $\Longleftrightarrow$ no redundant vectors

2. Linear dependence

   $\Longleftrightarrow$ Some redundant vector(s)

3. How to determine lin ind. or lin dep.

Linear Independence & Dependence

4. Basis $\{u_1, u_2, \ldots, u_n\}$ for V
   (1) lin ind, i.e. no redundancy
   (2) can span V

5. How to find basis vectors?

6. $\dim(V)$ = no. of vectors in a basis.
   (1) min. no. of vectors that can span V
   (2) max no. of lin. ind. vectors in V

$A^{m\times n}$, $Col(A)$, $Null(A)$, $Row(A)$

(1)

- $Col(A) = \text{Span}\{\text{cols of } A\}$
- How to find basis vector
- $\dim(Col(A)) = \#$ of pivots $= Rank(A)$

(2)

- $Null(A) = \{X : AX = 0\}$
- How to find basis vectors
- $\dim(Null(A)) = \#$ of free va. $= Nullity(A)$

$A^{m\times n}$, $Col(A)$, $Null(A)$, $Row(A)$

(3)

- $Row(A) = Span\{\text{rows of } A\}$
- How to find basis vector
- $dim(Row(A)) = \#$ of pivots $= Rank(A)$

(4) Rank - Nullity Theorem

$$\boxed{Rank(A) + Nullity(A) = n}$$

$A^{m\times n}$, $\mathrm{Col}(A)$, $\mathrm{Null}(A)$, $\mathrm{Row}(A)$

(5)

- $\mathrm{Row}(A) = \mathrm{Span}\{\text{rows of } A\}$
- How to find basis vector
- $\dim(\mathrm{Row}(A)) = \#$ of pivots $= \mathrm{Rank}(A)$

(6) Rank - Nullity Theorem

$$\boxed{\text{no. of pivots} + \text{no. of free} = \text{Total no. of variables}}$$

When Does A^{-1} exist? ($A^{n\times n}$)

A^{-1} exists is equivalent to any of the following:

(1) A is onto

(2) A is one-to-one

(3) col's of A are lin ind

(4) row's of A are lin ind

(5) $AX=b$ is solvable for any b.

(6) $AX=b$ has unique (at most one) solution

When Does A^{-1} exist? ($A^{n\times n}$)

A^{-1} exists is equivalent to any of the following:

(7) $\text{Rank}(A) = n$

(8) $\text{Nullity}(A) = 0$

(9) all variables are pivots

(10) no free variables

(11) $\det(A) \neq 0$

(12) $\lambda = 0$ is not an eigenvalue

When Does A^{-1} exist? ($A^{n\times n}$)

① Finding A^{-1}

$$(A \mid I) \to \to \cdots \to (I \mid B)$$

A^{-1}

② If $AB = I$, then $BA = I$

Linear Transformation $T: U \longrightarrow V$

(1) $T(\alpha u_1 + \beta u_2) = \alpha T(u_1) + \beta T(u_2)$

(2) Determine if T is onto

(3) Determine if T is one-to-one

(4) $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$, linear,

$$T(X) = AX$$

Linear Transformation $T: U \longrightarrow V$

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$$T(X) = AX$$

$A^{m \times n}$ is called the matrix of T.

Determinants

(1) Computation of $\det(A)$, $A^{n \times n}$

(a) by co-factor expansion

(b) by row reduction

(2) Properties of determinants

(3) $\boxed{\det(A) \neq 0 \Longleftrightarrow A^{-1} \text{ exists}}$

Determinants

(4) Formula for A^{-1} using determinants

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(5) Formula for solution of $AX=b$

(6) $\det(AB) = \det(A)\det(B)$

$$\det(A^T) = \det(A)$$

Eigenvalues (λ) & Eigenvectors (X)

(1) $AX = \lambda X$, $\boxed{X \neq 0}$

(2) $p(\lambda) = \det(A - \lambda I)$ = characteristic poly.

$$= c(\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_k)^{m_k}$$

m_i = algebraic multiplicity

g_i = geometric multiplicity = $\dim(\mathrm{Null}(A - \lambda I))$

= no. of lin. ind. eigenvectors for λ_i

= no. of free var. of $(A - \lambda I)X = 0$

Eigenvalues (λ) & Eigenvectors (X)

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(2) $p(\lambda) = \det(A - \lambda I)$ = characteristic poly.

$$= c(\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_k)^{m_k}$$

$$\boxed{1 \leq \underset{g_i}{\text{geom. multiplicity}} \leq \underset{m_i}{\text{algebraic multiplicity}}}$$

Eigenvalues (λ) & Eigenvectors (X)

(3) λ_i is defective/deficient if $g_i < m_i$,
non-defective/non-deficient if $g_i = m_i$

(4) For non-defective/non-deficient A,
it can be written as: $\boxed{A = QDQ^{-1}}$

(5) Application of $A = QDQ^{-1}$: to compute

$$\boxed{A^n = QD^nQ^{-1}}$$

Last Word.....

Notation Matters!!!