MA 351 (2024 Fall, Yip) Selected HW Solutions

Homework 2

p.63 #1.69

$$\left(\begin{array}{cccc|c} 2 & 4 & 1 & 3 & a \\ -3 & 1 & 2 & -2 & b \\ 13 & 5 & -4 & 12 & c \\ 12 & 10 & -1 & 13 & d \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 2 & 4 & 1 & 3 & a \\ -1 & 5 & 3 & 1 & a+b \\ 13 & 5 & -4 & 12 & c \\ 12 & 10 & -1 & 13 & d \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & -5 & -3 & -1 & -a-b \\ 2 & 4 & 1 & 3 & a \\ 13 & 5 & -4 & 12 & c \\ 12 & 10 & -1 & 13 & d \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & -5 & -3 & -1 & -a-b \\ 0 & 14 & 7 & 5 & 3a+2b \\ 0 & 70 & 35 & 25 & 13a+13b+c \\ 0 & 70 & 35 & 25 & 12a+12b+d \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & -5 & -3 & -1 & -a-b \\ 0 & 14 & 7 & 5 & 3a+2b \\ 0 & 70 & 35 & 25 & 13a+13b+c \\ 0 & 0 & 0 & 0 & -a-b+d-c \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & -5 & -3 & -1 & -a-b \\ 0 & 14 & 7 & 5 & 3a+2b \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & -a-b+d-c \end{array}\right)$$

$$13a+13b+c-5(3a+2b)$$

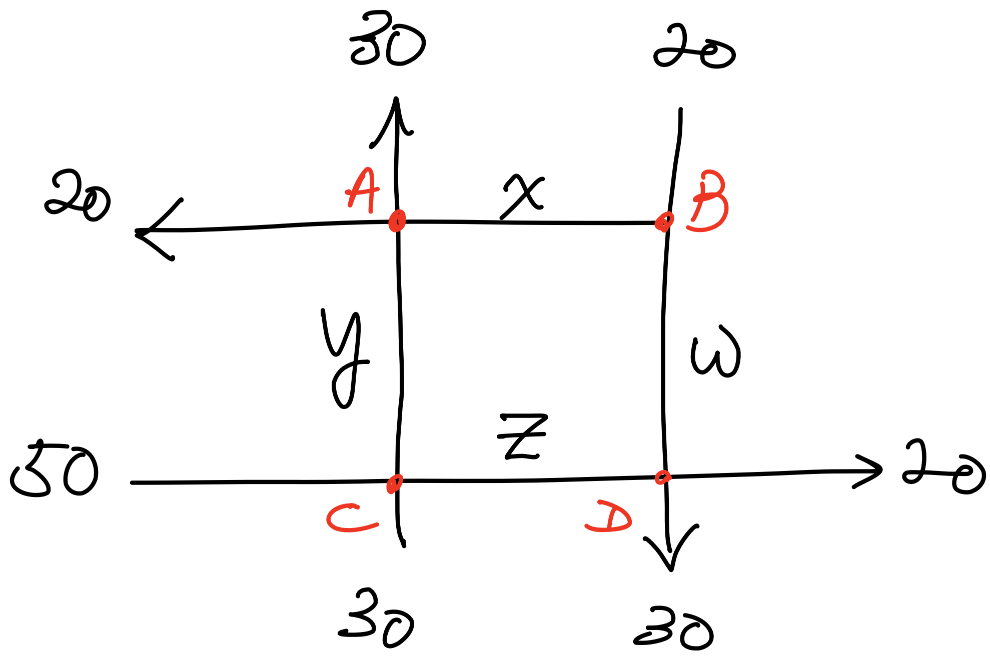
$$= -2a+3b+c$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & -5 & -3 & -1 & -a-b \\ 0 & 14 & 7 & 5 & 3a+2b \\ 0 & 0 & 0 & 0 & -2a+3b+c \\ 0 & 0 & 0 & 0 & -a-b+d-c \end{array}\right)$$

In order to have solution, we need:

$\underline{-2a+3b+c=0}$ and $\underline{-a-b+d-c=0}$

p.73 #1.90



A : $x + y = 20 + 30 = 50$

B : $x + w = 20$

C : $y + z = 50 + 30 = 80$

D : $z + w = 30 + 20 = 50$

$$\begin{array}{cccc} x & y & z & w \end{array}$$
$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 50 \\ 1 & 0 & 0 & 1 & 20 \\ 0 & 1 & 1 & 0 & 80 \\ 0 & 0 & 1 & 1 & 50 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 50 \\ 0 & -1 & 0 & 1 & -30 \\ 0 & 1 & 1 & 0 & 80 \\ 0 & 0 & 1 & 1 & 50 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 50 \\ 0 & 1 & 0 & -1 & 30 \\ 0 & 0 & 1 & 1 & 50 \\ 0 & 0 & 1 & 1 & 50 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 50 \\ 0 & 1 & 0 & -1 & 30 \\ 0 & 0 & 1 & 1 & 50 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 20 \\ 0 & 1 & 0 & -1 & 30 \\ 0 & 0 & 1 & 1 & 50 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

w (free)

$$\boxed{x = 20 - w,\ y = w + 30,\ z = 50 - w}$$

(a) $6 \le w \le 8$

$\Rightarrow$ $12 \le x \le 14$

$36 \le y \le 38$

$42 \le z \le 44$ $\leftarrow$ greatest traffic.

(b) $z \ge y \Rightarrow 50 - w \ge w + 30$

$\Rightarrow 10 \ge w$

$z \ge x \Rightarrow 50 - w \ge 20 - w$

$(50 \ge 20)$ ✓

Hence $0 \le w \le 10$

$\uparrow$

(one way street)

(c) Set $z = 0$

$\Rightarrow w = 50$

$\Rightarrow x = 20 - 50 = -30$

cannot have negative traffic in one-way street.

Additional Problem #1

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 1 & 2 & 4 & 3 \\ 1 & 3 & a & b \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & a-3 & b-2 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & a-5 & b-4 \end{array}\right)$$

(i) Unique solution: $\underline{a-5 \neq 0}$

(ii) Infinitely many solution:

$a-5=0$ & $b-4=0$, i.e. $\underline{a=5,\ b=4}$

(iii) No solution: $a-5=0$ but $b-4 \neq 0$

i.e. $\underline{a=5 \text{ but } b \neq 4}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 3 \\ 1 & 1 & c^2-5 & c \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & c^2-6 & c-2 \end{array}\right)$$

(i) Unique solution: $c^2-6 \neq 0$

(ii) Infinitely many solution:

$c^2-6=0$ & $c=2$ (Not possible)

(iii) No Solution:

$c^2-6=0$ but $c-2\neq 0$

$$\to c = \pm\sqrt{6}$$

Additional Problem #2

Let x, y, z be the number of units of A, M, S to be produced.

$$\begin{cases} x = 0.1x + 0.1y + 0.2z + 30 \\ y = 0.2x + 0.3y + 0.1z + 40 \\ z = 0.1x + 0.2y + 0.3z + 50 \end{cases}$$

$$\begin{cases} 0.9x - 0.1y - 0.2z = 30 \\ -0.2x + 0.7y - 0.1z = 40 \\ -0.1x - 0.2y + 0.7z = 50 \end{cases}$$

$$\left[\begin{array}{ccc|c} 0.9 & -0.1 & -0.2 & 30 \\ -0.2 & 0.7 & -0.1 & 40 \\ -0.1 & -0.2 & 0.7 & 50 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 9 & -1 & -2 & 300 \\ -2 & 7 & -1 & 400 \\ -1 & -2 & 7 & 500 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -7 & -500 \\ 9 & -1 & -2 & 300 \\ -2 & 7 & -1 & 400 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -7 & -500 \\ 0 & -19 & 61 & 4800 \\ 0 & 11 & -15 & -600 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -7 & -500 \\ 0 & 1 & -\frac{61}{19} & -4800/19 \\ 0 & 1 & -\frac{15}{11} & -\frac{600}{11} \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -7 & -500 \\ 0 & 1 & -\frac{61}{19} & -\frac{4800}{19} \\ 0 & 0 & \frac{61}{19} - \frac{15}{11} & \frac{4800}{19} - \frac{600}{11} \end{array}\right]$$

$$z = \left(\frac{4800}{19} - \frac{600}{11}\right) \Big/ \left(\frac{61}{19} - \frac{15}{11}\right) \cong 107.25$$

$$y = -\frac{4800}{19} + \frac{61}{19} \times 107.25 \cong 91.71$$

$$x = -500 - 2y + 7z \cong 67.33$$

p. 88 # 1.109

Homework 3

$$X_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix};$$

$$Y_1 = \begin{pmatrix} 2 \\ 3 \\ 2 \\ 2 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad Y_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\mathrm{Span}\{X_1, X_2, X_3\} \overset{?}{=} \mathrm{Span}\{Y_1, Y_2, Y_3\}$$

Try to express Y_1, Y_2, Y_3 in terms of lin. comb. of X_1, X_2, X_3 and vice versa.

$$\left[X_1 \; X_2 \; X_3 \;\middle|\; Y_1 \; Y_2 \; Y_3 \right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 2 & 0 & 1 \\ 2 & 1 & 0 & 3 & 1 & 1 \\ 1 & 1 & 1 & 2 & 0 & 1 \\ 1 & 1 & 2 & 2 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 2 & 0 & 1 \\ 0 & -1 & -2 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Hence $\text{Span}\{Y_1, Y_2, Y_3\} \subseteq \text{Span}\{X_1, X_2, X_3\}$

$$\left[\begin{array}{ccc|ccc} Y_1 & Y_2 & Y_3 & X_1 & X_2 & X_3 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 2 & 1 & 0 \\ 2 & 0 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 1 & 2 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Not possible

So,

$$\mathrm{Span}\{X_1, X_2, X_3\} \not\subseteq \mathrm{Span}\{Y_1, Y_2, Y_3\}$$

Hence

$$\mathrm{Span}\{X_1, X_2, X_3\} \neq \mathrm{Span}\{Y_1, Y_2, Y_3\}$$

1.110

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s_1 \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} + t_1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + s_2 \begin{pmatrix} -4 \\ 1 \\ 2 \end{pmatrix} + t_2 \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$$

Given $s_1, t_1 \Longrightarrow$ find s_2, t_2 (?)

$$s_2 \begin{pmatrix} -4 \\ 1 \\ 2 \end{pmatrix} + t_2 \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + s_1 \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} + t_1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -3s_1 - t_1 \\ 1 & + s_1 \\ -1 & + s_1 + t_1 \end{pmatrix}$$

$$\left[\begin{array}{cc|c} -4 & -3 & -3s_1 - t_1 \\ 1 & 1 & 1 + s_1 \\ 2 & 1 & -1 + s_1 + t_1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 + s_1 \\ -4 & -3 & -3s_1 - t_1 \\ 2 & 1 & -1 + s_1 + t_1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 + s_1 \\ 0 & 1 & 4 + s_1 - t_1 \\ 0 & -1 & -3 - s_1 + t_1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 + s_1 \\ 0 & 1 & 4 + s_1 - t_1 \\ 0 & 0 & 1 \end{array}\right] \leftarrow \text{not possible}$$

Hence the 2 representations are not the same.

(No need to check that given s_2, t_2, find s_1, t_1)

p. 89 #1.112 $(Z = 2X + 3Y)$

(a) $\text{Span}\{X, Y, Z\} = \text{Span}\{X, Y\}$

It is clear that

$$\text{Span}\{X, Y\} \subseteq \text{Span}\{X, Y, Z\}$$

$$aX + bY = aX + bY + 0 \cdot Z$$

Now

$$aX + bY + cZ = aX + bY + c(2X + 3Y)$$

$$= (a + 2c)X + (b + 3c)Y$$

Hence $\text{Span}\{X, Y, Z\} \subseteq \text{Span}\{X, Y\}$

(b) $\text{Span}\{X, Y, Z\} = \text{Span}\{X, Z\}$

Again, clear that

$$\text{Span}\{X, Z\} \subseteq \text{Span}\{X, Y, Z\}$$

Now

$$aX+bY+cZ$$

$$= aX+b\left(\frac{Z-2X}{3}\right)+cZ$$

$$= \left(a-\frac{2b}{3}\right)X+\left(\frac{b}{3}+c\right)Z$$

Hence $\mathrm{Span}\{X,Y,Z\} \subseteq \mathrm{Span}\{X,Z\}$

p. 92 #1.121

We have $AX=B$ and $AY=B$

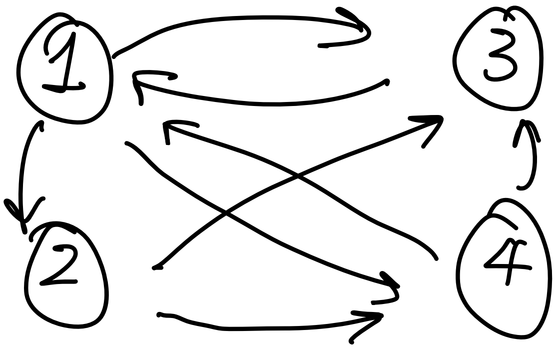
So $A(aX+bY) = aAX+bAY$

$$= aB+bB$$

$$= (a+b)B$$

Hence $a+b$ must be 1 (since $B\neq 0$) in order to have $A(aX+bY)=B$.

Additional Problem



$$\begin{cases} x_1 = x_3 + \dfrac{x_4}{2} \\ x_2 = \dfrac{1}{3}x_1 \\ x_3 = \dfrac{1}{3}x_1 + \dfrac{x_2}{2} + \dfrac{x_4}{2} \\ x_4 = \dfrac{1}{3}x_1 + \dfrac{x_2}{2} \end{cases}$$

$$\begin{cases} x_1 - x_3 - \dfrac{x_4}{2} = 0 \\ -\dfrac{1}{3}x_1 + x_2 = 0 \\ -\dfrac{1}{3}x_1 - \dfrac{x_2}{2} + x_3 - \dfrac{x_4}{2} = 0 \\ -\dfrac{1}{3}x_1 - \dfrac{x_2}{2} + x_4 = 0 \end{cases}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & -\frac{1}{2} & 0 \\ -\frac{1}{3} & 1 & 0 & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ -\frac{1}{3} & -\frac{1}{2} & 0 & 1 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & -1 & -\frac{1}{2} & 0 \\ 1 & -3 & 0 & 0 & 0 \\ 1 & +\frac{3}{2} & -3 & \frac{3}{2} & 0 \\ 1 & \frac{3}{2} & 0 & -3 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & -1 & -\frac{1}{2} & 0 \\ 0 & -3 & 1 & \frac{1}{2} & 0 \\ 0 & \frac{3}{2} & -2 & 2 & 0 \\ 0 & \frac{3}{2} & 1 & -\frac{5}{2} & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & -1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{3} & -\frac{1}{6} & 0 \\ 0 & 1 & -\frac{4}{3} & \frac{4}{3} & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{5}{3} & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{3} & -\frac{1}{6} & 0 \\ 0 & 0 & -1 & \frac{3}{2} & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{3} & -\frac{1}{6} & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_3 = \frac{3}{2}t \qquad x_4 = t$$

$$x_2 = \frac{1}{3}x_3 + \frac{1}{6}x_4 = \left(\frac{1}{2} + \frac{1}{6}\right)t = \frac{2}{3}t$$

$$x_1 = x_3 + \frac{1}{2}x_4 = \frac{3}{2}t + \frac{1}{2}t = 2t$$

$$\boxed{\begin{array}{ccccccc} x_1 & > & x_3 & > & x_4 & > & x_2 \\ (2t) & & \left(\frac{3}{2}t\right) & & (t) & & \left(\frac{2}{3}t\right) \end{array}}$$

Homework 4

p.20 #26

$$\underset{\vec{u}}{\begin{bmatrix}6\\1\\2\end{bmatrix}},\quad \underset{\vec{v}}{\begin{bmatrix}6\\2\\1\end{bmatrix}},\quad \underset{\vec{w}}{\begin{bmatrix}4\\1\\1\end{bmatrix}}$$

Consider $c_1\vec{u} + c_2\vec{v} + c_3\vec{w} = \vec{0}$

$$\left[\begin{array}{ccc|c} 6 & 6 & 4 & 0 \\ 1 & 2 & 1 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 6 & 6 & 4 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right] \begin{array}{l} R_2 \\ R_1 \\ R_3 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -6 & -2 & 0 \\ 0 & -3 & -1 & 0 \end{array}\right] \begin{array}{l} R_2 \\ R_1 - 6R_2 \\ R_3 - 2R_2 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1/3 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$c_1 = -\frac{\alpha}{3}, \quad c_2 = -\frac{\alpha}{3}, \quad c_3 = \alpha$$

Hence

$$-\frac{1}{3}\vec{u} - \frac{1}{3}\vec{v} + \vec{w} = \vec{0}$$

$$\Rightarrow \quad \vec{w} = \frac{1}{3}(\vec{u} + \vec{v}),$$

or $\quad \vec{u} = -\vec{v} + 3\vec{w}$

or $\quad \vec{v} = -\vec{u} + 3\vec{w}$

1.27 From the row reduction of the matrix, we see that

$$R_1 - 6R_2 = 2(R_3 - 2R_2)$$

i.e. $\boxed{R_1 - 2R_2 - 2R_3 = 0}$ i.e. $R_1 = 2R_2 + 2R_3$

Alternatively, make rows of the matrix as columns.

$$\begin{array}{cccc} R_1 & R_2 & R_3 & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 6 & 1 & 2 & 0 \\ 6 & 2 & 1 & 0 \\ 4 & 1 & 1 & 0 \end{array}\right] \to \begin{bmatrix} 1 & \frac{1}{6} & \frac{1}{3} \\ 1 & \frac{1}{3} & \frac{1}{6} \\ 4 & 1 & 1 \end{bmatrix}$$

$$\to \left[\begin{array}{ccc|c} 1 & 1/6 & 1/3 & 0 \\ 0 & 1/6 & -1/6 & 0 \\ 0 & 1/3 & -1/3 & 0 \end{array}\right]$$

$\frac{1}{3} - \frac{1}{6} =$

$$\to \left[\begin{array}{ccc|c} 1 & \frac{1}{6} & \frac{1}{3} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rightarrow \begin{array}{ccc|c} C_1 & C_2 & C_3 & \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$C_3 = \alpha, \quad C_2 = \alpha, \quad C_1 = -\frac{\alpha}{2}$

ie. $-\frac{\alpha}{2} R_1 + \alpha R_2 + \alpha R_3 = 0$

$\alpha = 1$

ie. $\boxed{R_1 = 2R_2 + 2R_3}$,

1.32 $S = \{A, B, C, D\}$

$$A = B - 3C + D, \quad C = A - B$$

(a) Consider $A = B - 3(A - B) + D$

$$\implies 4A = 4B + D$$

$$\implies 4A - 4B - D = 0$$

So $\{A, B, D\}$ is lin. dep.

(b) Consider $A = (A - C) - 3C + D$

$$\implies 4C = D$$

$$\rightarrow 0 \cdot A + 4C - D = 0$$

So $\{A, B, C\}$ is lin dep.

(c) Nothing can be said about lin dep/ind between A, C.

Practice Problems for Sec. 2.1

p.110 #2.3(d)

$$\begin{array}{c} \\ (-1)\times \\ \\ \\ \end{array}\begin{array}{c}\begin{bmatrix} 2 & 3 & -1 & 19 & 2 \\ -1 & 0 & 2 & -8 & 5 \\ 2 & -1 & -5 & 15 & -14 \end{bmatrix} \\ \begin{matrix} u_1 & u_2 & u_3 & u_4 & u_5 \end{matrix}\end{array}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & -2 & 8 & -5 \\ 2 & 3 & -1 & 19 & 2 \\ 2 & -1 & -5 & 15 & -14 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & -2 & 8 & -5 \\ 0 & 3 & 3 & 3 & 12 \\ 0 & -1 & -1 & -1 & -4 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & -2 & 8 & -5 \\ 0 & 1 & 1 & 1 & 4 \\ 0 & 3 & 3 & 3 & 12 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & -2 & 8 & -5 \\ 0 & 1 & 1 & 1 & 4 \\ 0 & 0 & \underbrace{0 \quad\; 0 \quad\; 0}_{c_3,\, c_4,\, c_5 \text{ - free}} \end{bmatrix}$$

$c_3 = \alpha, \quad c_4 = \beta, \quad c_5 = \gamma$

$c_1 = 2\alpha - 8\beta + 5\gamma$

$c_2 = -\alpha - \beta - 4\gamma$

The dependency relation between u_1, u_2, u_3, u_4 & u_5 is :

$$\boxed{(2\alpha - 8\beta + 5\gamma)\vec{u}_1 + (-\alpha - \beta - 4\gamma)\vec{u}_2 + \alpha\vec{u}_3 + \beta\vec{u}_4 + \gamma\vec{u}_5 = 0}$$

$\alpha = 1, \beta = 0, \gamma = 0 \Rightarrow \quad u_3 = -2u_1 + u_2$

$\alpha = 0, \beta = 1, \gamma = 0 \Rightarrow \quad u_4 = 8u_1 + u_2$

$\alpha = 0, \beta = 0, \gamma = 1 \Rightarrow \quad u_5 = -5u_1 + 4u_2$

p. 110 #27

$$A = \begin{bmatrix} 1 & a & b & c & d & e \\ 0 & 0 & 1 & f & g & h \\ 0 & 0 & 0 & 0 & 1 & k \end{bmatrix} \begin{array}{l} \leftarrow R_1 \\ \leftarrow R_2 \\ \leftarrow R_3 \end{array}$$

Consider $c_1R_1 + c_2R_2 + c_3R_3 = 0$

$$\begin{aligned} & c_1(1 \; a \; b \; c \; d \; e) \\ +\, & c_2(0 \; 0 \; 1 \; f \; g \; h) \\ +\, & c_3(0 \; 0 \; 0 \; 0 \; 1 \; k) \\ =\, & (0 \; 0 \; 0 \; 0 \; 0 \; 0) \end{aligned}$$

$\underline{c_1 = 0}$

$\underset{0}{\cancel{c_1}}\, b + c_2 = 0 \Downarrow \underline{c_2 = 0}$

$\underset{0}{\cancel{c_1}}\, d + \underset{0}{\cancel{c_2}}\, g + c_3 = 0 \Downarrow \underline{c_3 = 0}$

Hence R_1, R_2, R_3 are lin. ind.

p. 111 # 2.11

$$Y_1 = 3X_1 - 2X_2$$

$$Y_2 = X_1 + X_2$$

Consider $c_1 Y_1 + c_2 Y_2 = \vec{0}$

$$c_1(3X_1 - 2X_2) + c_2(X_1 + X_2) = \vec{0}$$

$$\underbrace{(3c_1 + c_2)}_{= 0} X_1 + \underbrace{(-2c_1 + c_2)}_{= 0} X_2 = \vec{0}$$

Since X_1, X_2 are lin ind.

$$\begin{cases} 3c_1 + c_2 = 0 \\ -2c_1 + c_2 = 0 \end{cases} \Longrightarrow \text{Solve for } c_1, c_2$$

$$c_1 = 0, \; c_2 = 0$$

Hence Y_1, Y_2 are lin. ind.

p.111 #2.13

$$Y_1 = X_1 + 2X_2 - X_3$$

$$Y_2 = X_1 + X_2$$

$$Y_3 = 7X_2$$

Consider $c_1 Y_1 + c_2 Y_2 + c_3 Y_3$

$$= c_1(X_1 + 2X_2 - X_3) + c_2(X_1 + X_2) + c_3(7X_1)$$

$$= (c_1 + c_2 + 7c_3)X_1 + (2c_1 + c_2)X_2 - c_1 X_3 = 0$$

Since X_1, X_2, X_3 are lin. ind, we must have

$$\begin{aligned} c_1 + c_2 + 7c_3 &= 0 \qquad c_3 = 0 \\ 2c_1 + c_2 &= 0 \qquad c_2 = 0 \\ -c_1 &= 0 \end{aligned}$$

Hence Y_1, Y_2, Y_3 are lin. ind.

p. 111 #2.17

$$Y_1 = X_1 + 2X_2 - X_3$$
$$Y_2 = 2X_1 + 2X_2 - X_3$$
$$Y_3 = 4X_1 + 2X_2 - X_3$$

Consider $c_1 Y_1 + c_2 Y_2 + c_3 Y_3 = \vec{0}$

$$c_1(X_1 + 2X_2 - X_3) + c_2(2X_1 + 2X_2 - X_3) + c_3(4X_1 + 2X_2 - X_3) = \vec{0}$$

$$(c_1 + 2c_1 + 4c_3)X_1 + (2c_1 + 2c_2 + 2c_3)X_2 + (-c_1 - c_2 - c_3)X_3 = 0$$

Look for c_1, c_2, c_3 s.t.

$$\begin{cases} c_1 + 2c_1 + 4c_3 = 0 \\ 2c_1 + 2c_2 + 2c_3 = 0 \\ -c_1 - c_2 - c_3 = 0 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 2 & 2 & 2 & 0 \\ -1 & -1 & -1 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & -1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \begin{array}{l} c_1 = 2c_3 \\ c_2 = -3c_3 \end{array}$$

It is always true that

$$2Y_1 - 3Y_2 + Y_3 = \vec{0}$$

Y_1, Y_2, Y_3 are lin dep.

regardless if X_1, X_2, X_3 are lin. ind. or not.

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$$Y_1 = 2X_1 + 3X_2 + 5X_3$$
$$Y_2 = X_1 - X_2 + 3X_3$$
$$Y_3 = 2X_1 + 13X_2 + 3X_3$$

Look for c_1, c_2, c_3 s.t. $c_1 Y_1 + c_2 Y_2 + c_3 Y_3 = \vec{0}$

i.e. $c_1(2X_1 + 3X_2 + 5X_3)$
$+ c_2(X_1 - X_2 + 3X_3)$
$+ c_3(2X_1 + 13X_2 + 3X_3) = \vec{0}$

$(2c_1 + c_2 + 2c_3)X_1 + (3c_1 - c_2 + 13c_3)X_2$
$+ (5c_1 + 3c_2 + 3c_3)X_3 = \vec{0}$

Set
$$\begin{cases} 2c_1 + c_2 + 2c_3 = 0 \\ 3c_1 - c_2 + 13c_3 = 0 \\ 5c_1 + 3c_2 + 3c_3 = 0 \end{cases}$$

$$\left(\begin{array}{ccc|c} 2 & 1 & 2 & 0 \\ 3 & -1 & 13 & 0 \\ 5 & 3 & 3 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 2 & 1 & 2 & 0 \\ 1 & -2 & 11 & 0 \\ 5 & 3 & 3 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & -2 & 11 & 0 \\ 2 & 1 & 2 & 0 \\ 5 & 3 & 3 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & -2 & 11 & 0 \\ 0 & 5 & -20 & 0 \\ 0 & 13 & -52 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & -2 & 11 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{array}{l} C_1 = -3C_3 \\ C_2 = 4C_3 \end{array}$$

Hence we always have

$$-3Y_1 + 4Y_2 + Y_3 = 0$$

Y_1, Y_2, Y_3 are lin dep. regardless if X_1, X_2, X_3 are lin ind or not.