Homework 10 Selected Solution

4.4 Use expansion along the third row to express each of the following determinants as a sum of determinants of 4×4 matrices. Without expanding further, explain why $3\alpha = \beta$. What theorem from the text does this exercise demonstrate?

$$\alpha = \begin{vmatrix} 24 & -13 & 7 & 9 & 5 \\ 11 & 16 & -37 & 99 & 64 \\ 1 & 4 & 2 & 2 & -3 \\ 31 & -42 & 78 & 55 & -3 \\ 62 & 47 & 29 & -14 & -8 \end{vmatrix}, \quad \beta = \begin{vmatrix} 24 & -13 & 7 & 9 & 5 \\ 11 & 16 & -37 & 99 & 64 \\ 3 & 12 & 6 & 6 & -9 \\ 31 & -42 & 78 & 55 & -3 \\ 62 & 47 & 29 & -14 & -8 \end{vmatrix}$$

$$3(1 \quad 4 \quad 2 \quad 2 \quad -3) = (3 \quad 12 \quad 6 \quad 6 \quad -9)$$

4.5 ✓✓Use expansion along the third row to express β, δ, and γ as a sum of determinants of 4×4 matrices, where β is as in Exercise 4.4 and δ and γ are as follows. Without expanding further, explain why $\beta = \delta + \gamma$. What theorem from the text does this exercise demonstrate?

$$\delta = \begin{vmatrix} 24 & -13 & 7 & 9 & 5 \\ 11 & 16 & -37 & 99 & 64 \\ 1 & 7 & 3 & 3 & -13 \\ 31 & -42 & 78 & 55 & -3 \\ 62 & 47 & 29 & -14 & -8 \end{vmatrix}, \quad \gamma = \begin{vmatrix} 24 & -13 & 7 & 9 & 5 \\ 11 & 16 & -37 & 99 & 64 \\ 2 & 5 & 3 & 3 & 4 \\ 31 & -42 & 78 & 55 & -3 \\ 62 & 47 & 29 & -14 & -8 \end{vmatrix}$$

$$(3 \quad 12 \quad 6 \quad 6 \quad -9) = (1 \quad 7 \quad 3 \quad 3 \quad -13) + (2 \quad 5 \quad 3 \quad 3 \quad 4)$$

Det is alternating multilinear function on the rows and columns of a matrix

4.15 ✓✓Suppose that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 5$$

Compute the following determinants:

(a) $\begin{vmatrix} 2a & 2b & 2c \\ 3d-a & 3e-b & 3f-c \\ 4g+3a & 4h+3b & 4i+3c \end{vmatrix}$ **(b)** $\begin{vmatrix} a+2d & b+2e & c+2f \\ g & h & i \\ d & e & f \end{vmatrix}$

$$(a) = 2\begin{vmatrix} a & b & c \\ 3d-a & 3e-b & 3f-b \\ 4g+3a & 4h+3b & 4i+3c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ 3d & 3e & 3f \\ 4g & 4h & 4i \end{vmatrix}$$

$$= 2(3)(4)\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 2(3)(4)(5) = \underline{120}$$

$$(b) = \begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix} = -\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -5$$

4.16 $$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = \begin{vmatrix} \boxed{1} & 1 & 1 \\ 0 & y-x & z-x \\ 0 & y^2-x^2 & z^2-x^2 \end{vmatrix} = \begin{vmatrix} y-x & z-x \\ y^2-x^2 & z^2-x^2 \end{vmatrix}$$

$$= (y-x)(z-x)\begin{vmatrix} 1 & 1 \\ y+x & z+x \end{vmatrix} = (y-x)(z-x)(z-y)$$

$$\det(AB) = (\det A)(\det B), \quad \det(A^T) = \det A$$

p.260 4.24

$$AA^{-1} = I$$
$$\det(AA^{-1}) = \det I$$
$$(\det A)(\det A^{-1}) = 1$$
$$\det(A^{-1}) = \frac{1}{\det A}$$

#4.25

$$A = QBQ^{-1}$$
$$\det(A) = \det(QBQ^{-1})$$
$$= (\det Q)(\det B)(\det Q^{-1})$$

$(\det Q)(\det Q^{-1}) = 1$

$$= \det B$$

#4.26

$$AA^T = I$$
$$\det(AA^T) = \det(I)$$
$$(\det A)(\det(A^T)) = 1$$

$(\det A = \det A^T)$

$$(\det A)^2 = 1 \implies (\det A) = \pm 1$$

Homework 11 Selected Solution

4.36 Use Cramer's rule to solve the following system for z in terms of p_1, p_2, and p_3. (Note that we have not asked for x or y.)

$$\begin{bmatrix} 1 & 2 & -3 \\ 3 & 1 & -1 \\ 2 & 3 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \begin{aligned} x + 2y - 3z &= p_1 \\ 3x + y - z &= p_2 \\ 2x + 3y + 5z &= p_3 \end{aligned}$$

(1)

$$x = \begin{vmatrix} p_1 & 2 & -3 \\ p_2 & 1 & -1 \\ p_3 & 3 & 5 \end{vmatrix} \Bigg/ \begin{vmatrix} 1 & 2 & -3 \\ 3 & 1 & -1 \\ 2 & 3 & 5 \end{vmatrix} = \frac{p_1(8) - p_2(19) + p_3(1)}{(-47)}$$

$$y = \begin{vmatrix} 1 & p_1 & -3 \\ 3 & p_2 & -1 \\ 2 & p_3 & 5 \end{vmatrix} \Bigg/ \begin{vmatrix} 1 & 2 & -3 \\ 3 & 1 & -1 \\ 2 & 3 & 5 \end{vmatrix} = \frac{-p_1(17) + p_2(11) - p_3(8)}{(-47)}$$

$$z = \begin{vmatrix} 1 & 2 & p_1 \\ 3 & 1 & p_2 \\ 2 & 3 & p_3 \end{vmatrix} \Bigg/ \begin{vmatrix} 1 & 2 & -3 \\ 3 & 1 & -1 \\ 2 & 3 & 5 \end{vmatrix} = \frac{p_1(7) - p_2(-1) + p_3(-5)}{(-47)}$$

(2)

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & p_1 \\ 3 & 1 & -1 & p_2 \\ 2 & 3 & 5 & p_3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & -3 & p_1 \\ 0 & -5 & 8 & p_2 - 3p_1 \\ 0 & -1 & 11 & p_3 - 2p_1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 2 & -3 & p_1 \\ 0 & 1 & -11 & 2p_1 - p_3 \\ 0 & 5 & -8 & 3p_1 - p_2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & -3 & p_1 \\ 0 & 1 & -11 & 2p_1 - p_3 \\ 0 & 0 & 47 & \end{array}\right)$$

$$-7p_1 - p_2 + 5p_3$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -3 & p_1 \\ 0 & 1 & -11 & 2p_1 - p_3 \\ 0 & 0 & 1 & -\frac{7}{47}p_1 - \frac{p_2}{47} + \frac{5}{47}p_3 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 0 & \frac{26}{47}p_1 - \frac{3}{47}p_2 + \frac{15}{47}p_3 \\ 0 & 1 & 0 & \frac{17}{47}p_1 - \frac{11}{47}p_2 + \frac{8}{47}p_3 \\ 0 & 0 & 1 & -\frac{7}{47}p_1 - \frac{p_2}{47} + \frac{5}{47}p_3 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{8}{47}p_1 + \frac{19}{47}p_2 - \frac{1}{47}p_3 \\ 0 & 1 & 0 & \frac{17}{47}p_1 - \frac{11}{47}p_2 + \frac{8}{47}p_3 \\ 0 & 0 & 1 & -\frac{7}{47}p_1 - \frac{p_2}{47} + \frac{5}{47}p_3 \end{array}\right)$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{8}{47}p_1 + \frac{19}{47}p_2 - \frac{1}{47}p_3 \\ \frac{17}{47}p_1 - \frac{11}{47}p_2 + \frac{8}{47}p_3 \\ -\frac{7}{47}p_1 - \frac{p_2}{47} + \frac{5}{47}p_3 \end{pmatrix} = \underbrace{\frac{1}{47}\begin{pmatrix} -8 & 19 & -1 \\ 17 & -11 & 8 \\ -7 & -1 & 5 \end{pmatrix}}_{A^{-1}} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

(3) $\begin{pmatrix}1&2&-3\\3&1&-1\\2&3&5\end{pmatrix}^{-1}$

$$= \begin{pmatrix} \begin{vmatrix}1&-1\\3&5\end{vmatrix} & -\begin{vmatrix}3&-1\\2&5\end{vmatrix} & \begin{vmatrix}3&1\\2&3\end{vmatrix}\\ -\begin{vmatrix}2&-3\\3&5\end{vmatrix} & \begin{vmatrix}1&-3\\2&5\end{vmatrix} & -\begin{vmatrix}1&2\\2&3\end{vmatrix} \\ \begin{vmatrix}2&-3\\1&-1\end{vmatrix} & -\begin{vmatrix}1&-3\\3&-1\end{vmatrix} & \begin{vmatrix}1&2\\3&1\end{vmatrix}\end{pmatrix}^T \Bigg/ \begin{vmatrix}1&2&-3\\3&1&-1\\2&3&5\end{vmatrix}$$

$$\left(\begin{vmatrix}1&2&-3\\3&1&-1\\2&3&5\end{vmatrix} = \begin{vmatrix}1&2&-3\\0&-5&8\\0&-1&11\end{vmatrix} = -55+8 = -47 \right)$$

$$= \begin{pmatrix} 8 & -17 & 7 \\ -19 & 11 & 1 \\ 1 & -8 & -5 \end{pmatrix}^T \left(-\frac{1}{47}\right)$$

$$= \frac{1}{47} \begin{pmatrix} -8 & 19 & -1 \\ 17 & -11 & 8 \\ -7 & -1 & 5 \end{pmatrix}$$

p. 280 #5.3 (a)

$$A = \begin{bmatrix} 0 & 3 & -3 \\ 2 & 2 & -2 \\ -4 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 3 & -3 \\ 2 & 2 & -2 \\ -4 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ -6 \end{bmatrix} = \overset{\lambda_1}{6} \underbrace{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}_{X_1}$$

$$\begin{bmatrix} 0 & 3 & -3 \\ 2 & 2 & -2 \\ -4 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ -3 \end{bmatrix} = \overset{\lambda_2}{-3} \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{X_2}$$

$$\begin{bmatrix} 0 & 3 & -3 \\ 2 & 2 & -2 \\ -4 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \overset{\lambda_3}{0} \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{X_3}$$

$$B = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = c_1 X_1 + c_2 X_2 + c_3 X_3$$

$$= 0 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Hence

$$A^n B = c_1\lambda_1^n X_1 + c_2\lambda_2^n X_2 + c_3\lambda_3^n X_3$$

$$= 0\times 6^n \begin{bmatrix}1\\1\\-1\end{bmatrix} + 1(-3)^n\begin{bmatrix}1\\0\\1\end{bmatrix} + 1(0)^n \begin{bmatrix}0\\1\\1\end{bmatrix}$$

$$= (-3)^n \begin{bmatrix}1\\0\\1\end{bmatrix}$$

#5.11 $A^2 = I$

$$AX = \lambda X$$

$\xrightarrow{A}$

$$\underbrace{A^2}_{= I} X = \lambda A X = \lambda^2 X$$

Hence $X = \lambda^2 X \Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1$

$(X \neq 0)$

#5.12 $AX = \lambda X$

$\xrightarrow{A}$

$$A^2 X = \lambda A X = \lambda^2 X$$

#5.13

$$AX = \lambda X$$

$A^{-1} \nearrow$

$$A^{-1}AX = \lambda A^{-1}X$$

$$IX = \lambda A^{-1}X$$

$$X = \lambda A^{-1}X$$

$$A^{-1}X = \frac{1}{\lambda}X$$

#5.14 $p(\lambda) = \det(A - \lambda I) = \lambda^2(\lambda+5)^3(\lambda-7)^5$

(a) A is 10×10

(b) A is not invertible as $\lambda = 0$ is an eigenvalue.

(If A^{-1} exists, $AX = 0X$

$\xrightarrow{A^{-1}} X = A^{-1}(0X) = \vec{0}$!!!

Thm: A^{-1} exists $\iff$ $\lambda = 0$ is not an eigenvalue

or equivalently,

A^{-1} does not exists $\iff$ $\lambda = 0$ is an eigenvalue)

(c) $Null(A) = \{X : AX = \vec{0}\}$

$= \{X : AX = 0X\}$

$=$ Eigenspace for $\lambda = 0$

Hence $1 \leq \dim(Null(A)) \leq 2$

geometric multiplicity of $\lambda = 0$

algebraic multiplicity of $\lambda = 0$

(d) $1 \leq \dim(\text{Eigenspace for } \lambda = 7) \leq 5$

Additional Problem

$$\underset{n\times n}{A}\,\underset{n\times n}{B} = \underset{n\times n}{I}$$

(1) $AX = Y$. Let $X = BY$. Check: $A\underbrace{(BY)}_{X} = (AB)Y = Y$

(2) $\mathrm{Rank}(A) = n \; (= \dim(\mathrm{Col}(A)) = \dim(\mathrm{Row}(A)))$

(3) $\mathrm{Rank}(A^T) = n \; (= \dim(\mathrm{Col}(A^T)) = \dim(\mathrm{Row}(A^T)))$

(4) $A^TX = Y$ is solvable for any Y as $\mathrm{Rank}(A^T) = n$

(5) $A^Tu_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $A^Tu_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$, $\cdots$ $A^Tu_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

$$A^T \underbrace{\begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}}_{C} = \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & 0 & & 0 \\ 0 & 0 & & 1 \end{bmatrix}}_{I}$$

i.e. $A^TC = I$

(6) $(A^TC)^T = I^T \implies C^TA = I$

(7) $AB = I$ and $\underline{C^TA = I}$

$C^TAB = C^TI \implies B = C^T \implies \underline{BA = I}$

Practice Problem for Chapter 5

#5.10 $F_1 = 1,\ F_2 = 1,\ F_{n+1} = F_n + F_{n-1},\ n \geq 2$

Let $Y_1 = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\quad Y_n = \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix}$

Then $Y_n = \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} F_n \\ F_n + F_{n-1} \end{bmatrix}$

$$= \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} F_{n-1} \\ F_n \end{bmatrix}}_{Y_{n-1}}$$

Hence $Y_n = AY_{n-1} = AAY_{n-2} = A^2 Y_{n-2}$

$$= A^3 Y_{n-3} \cdots\cdots$$

$$= A^{n-1} Y_1$$

To compute A^n

$\det(A - \lambda I) = \det\begin{pmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} = -\lambda(1-\lambda) - 1$

$= \lambda^2 - \lambda - 1 = 0$

$$\lambda = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$\boxed{\lambda_1 = \frac{1+\sqrt{5}}{2},} \qquad \left(A - \lambda_1 I \,\middle|\, 0\right)$$

$$\to \left(\begin{array}{cc|c} -\frac{1+\sqrt{5}}{2} & 1 & 0 \\ 1 & \frac{1-\sqrt{5}}{2} & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & \frac{1-\sqrt{5}}{2} & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$X_1 = \begin{pmatrix} \frac{\sqrt{5}-1}{2} \\ 1 \end{pmatrix} \qquad \boxed{\text{Check: } \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \frac{\sqrt{5}-1}{2} \\ 1 \end{pmatrix} = \frac{1+\sqrt{5}}{2}\begin{pmatrix} \frac{\sqrt{5}-1}{2} \\ 1 \end{pmatrix}}$$

$$\boxed{\lambda_2 = \frac{1-\sqrt{5}}{2},} \qquad \left(A - \lambda_2 I \,\middle|\, 0\right)$$

$$\to \left(\begin{array}{cc|c} -\frac{1-\sqrt{5}}{2} & 1 & 0 \\ 1 & \frac{1+\sqrt{5}}{2} & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & \frac{1+\sqrt{5}}{2} & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$X_2 = \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ -1 \end{pmatrix} \qquad \boxed{\text{Check: } \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ -1 \end{pmatrix} = \frac{1-\sqrt{5}}{2}\begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ -1 \end{pmatrix}}$$

Write $Y_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_1 X_1 + c_2 X_2$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} \frac{\sqrt{5}-1}{2} \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ -1 \end{bmatrix}$$

$$= \left(\frac{3+\sqrt{5}}{2\sqrt{5}}\right) \begin{bmatrix} \frac{\sqrt{5}-1}{2} \\ 1 \end{bmatrix} + \left(\frac{3-\sqrt{5}}{2\sqrt{5}}\right) \begin{bmatrix} \frac{\sqrt{5}+1}{2} \\ -1 \end{bmatrix}$$

Hence

$$A^{n-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} =$$

$$\left(\frac{3+\sqrt{5}}{2\sqrt{5}}\right) \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} \begin{pmatrix} \frac{\sqrt{5}-1}{2} \\ 1 \end{pmatrix} + \left(\frac{3-\sqrt{5}}{2\sqrt{5}}\right) \left(\frac{1-\sqrt{5}}{2}\right)^{n-1} \begin{pmatrix} \frac{\sqrt{5}+1}{2} \\ -1 \end{pmatrix}$$

& $$F_n = \left(\frac{3+\sqrt{5}}{2\sqrt{5}}\right) \left(\frac{\sqrt{5}-1}{2}\right) \left(\frac{1+\sqrt{5}}{2}\right)^{n-1}$$

$$+ \left(\frac{3-\sqrt{5}}{2\sqrt{5}}\right) \left(\frac{\sqrt{5}+1}{2}\right) \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}$$

$$= \underbrace{\left(\frac{3+\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{\sqrt{5}-1}{2}\right)\left(\frac{2}{1+\sqrt{5}}\right)}_{=\frac{1}{\sqrt{5}}}\left(\frac{1+\sqrt{5}}{2}\right)^{n}$$

$$+ \underbrace{\left(\frac{3-\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{\sqrt{5}+1}{2}\right)\left(\frac{2}{1-\sqrt{5}}\right)}_{=-\frac{1}{\sqrt{5}}}\left(\frac{1-\sqrt{5}}{2}\right)^{n}$$

$$= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n}$$

$$\boxed{F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n} - \left(\frac{1-\sqrt{5}}{2}\right)^{n}\right]}$$

#3.15 $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$p(\lambda) = \det(A - \lambda I) = \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$$

$$= (a-\lambda)(d-\lambda) - bc$$

$$= \lambda^2 - (a+d)\lambda + ad - bc$$

For $p(\lambda) = (\lambda-2)(\lambda-3) = \lambda^2 - 5\lambda + 6$

Find a, b, c, d s.t. $a+d = 5$
$ad - bc = 6$

$d = 5 - a$, $a(5-a) - bc = 6$
$a^2 - 5a + bc = -6$

eg1 $a=0$, $d=5$, $b=1$, $c=-6$

$$A = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}$$

eg2 $a=0$, $d=5$, $b=-1$, $c=6$

$$A = \begin{bmatrix} 0 & -1 \\ 6 & 5 \end{bmatrix}$$

#5.16 $p(\lambda) = \lambda^2 - (a+d)\lambda + (ad-bc) = 0$

Need $\Delta = (a+d)^2 - 4(ad-bc)$

$= a^2 + 2ad + d^2 - 4ad + 4bc$

$= (a-d)^2 + 4bc < 0$

Then no real roots.

p. 285 #5.7

(a) $A = \begin{bmatrix} 0.3 & 0.5 \\ 0.7 & 0.5 \end{bmatrix}$

Equilibrium vector X: $AX = X$ (Thm 3.3)

$(A-I)X = 0$

i.e. X is an eigenvector for eigenvalue 1

(X is a probability vector in the sense that the sum of all the entries equals 1.)

$$(A-I|\vec{0}) \to \left(\begin{array}{cc|c} -0.7 & 0.5 & 0 \\ 0.7 & -0.5 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cc|c} 0.7 & -0.5 & 0 \\ 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 7 & -5 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$X = \begin{pmatrix} 5\alpha \\ 7\alpha \end{pmatrix} = \begin{pmatrix} 5/12 \\ 7/12 \end{pmatrix} \quad (\alpha = \tfrac{1}{12})$$

$$\boxed{\text{Check: } \begin{pmatrix} 0.3 & 0.5 \\ 0.7 & 0.5 \end{pmatrix} \begin{pmatrix} 5/12 \\ 7/12 \end{pmatrix} = \begin{pmatrix} 5/12 \\ 7/12 \end{pmatrix}}$$

(b) $A = \begin{bmatrix} 0.1 & 0.3 & 0.2 \\ 0.4 & 0.7 & 0.1 \\ 0.5 & 0 & 0.7 \end{bmatrix}$

$$(A - I \mid \vec{0}) \rightarrow \left(\begin{array}{ccc|c} -0.9 & 0.3 & 0.2 & 0 \\ 0.4 & -0.3 & 0.1 & 0 \\ 0.5 & 0 & -0.3 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & -\frac{1}{3} & -\frac{2}{9} & 0 \\ 4 & -3 & 1 & 0 \\ 5 & 0 & -3 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & -1/3 & -2/9 & 0 \\ 0 & -5/3 & 17/9 & 0 \\ 0 & 5/3 & -17/9 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & -1/3 & -2/9 & 0 \\ 0 & 5/3 & -17/9 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

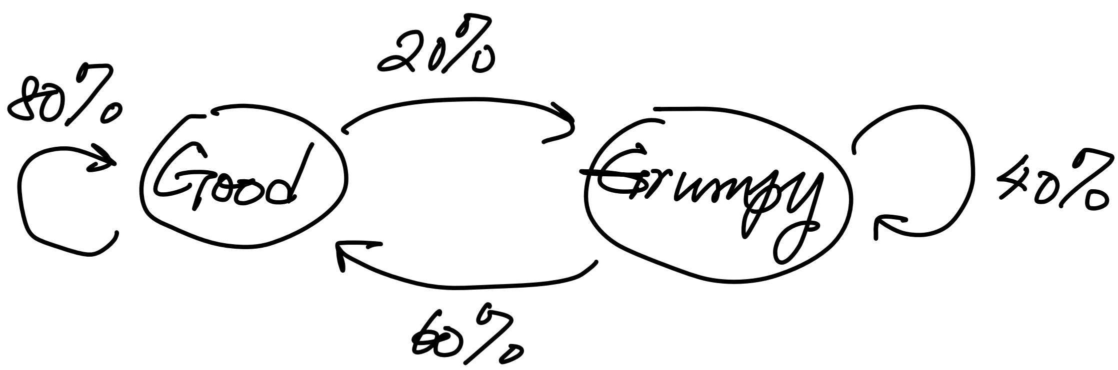
$$\rightarrow \left(\begin{array}{ccc|c} 1 & -1/3 & -2/9 & 0 \\ 0 & 1 & -17/15 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -9/15 & 0 \\ 0 & 1 & -17/15 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$x = \begin{pmatrix} \frac{9}{15}\alpha \\ \frac{17}{15}\alpha \\ \alpha \end{pmatrix} = \begin{pmatrix} 9/41 \\ 17/41 \\ 15/41 \end{pmatrix} \qquad \left(\alpha = \frac{15}{41}\right)$$

Check: $\begin{pmatrix} 0.1 & 0.3 & 0.2 \\ 0.4 & 0.7 & 0.1 \\ 0.5 & 0 & 0.7 \end{pmatrix} \begin{pmatrix} 9/41 \\ 17/41 \\ 15/41 \end{pmatrix} = \begin{pmatrix} 9/41 \\ 17/41 \\ 15/41 \end{pmatrix}$

#5.20



(a) $A = \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix}$

(b) Let $Y_n = \begin{pmatrix} \text{probability in good mood in day } n \\ \text{probability in grumpy mood in day } n \end{pmatrix}$

Then $Y_n = A^n Y_0$ ← initial state.

To find A^n. Diagonalize A

$$\det(A-\lambda I) = \det\begin{pmatrix} 0.8-\lambda & 0.6 \\ 0.2 & 0.4-\lambda \end{pmatrix}$$

$$= (0.8-\lambda)(0.4-\lambda) - 0.12$$
$$= \lambda^2 - 1.2\lambda + 0.32 - 0.12$$
$$= \lambda^2 - 1.2\lambda + 0.2$$
$$= (\lambda-1)(\lambda-0.2) = 0 \Rightarrow \lambda = 1,\ 0.2$$

$\lambda=1$: $(A-I|\vec{0}) \rightarrow \left(\begin{array}{cc|c} -0.2 & 0.6 & 0 \\ 0.2 & -0.6 & 0 \end{array}\right)$

$\rightarrow \left(\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow X = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\lambda=0.2$: $(A-0.2I|\vec{0}) \rightarrow \begin{pmatrix} 0.6 & 0.6 \\ 0.2 & 0.2 \end{pmatrix}$

$\rightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow X = \begin{pmatrix} +1 \\ -1 \end{pmatrix}$

Now let $Y_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\quad Y_0 = \frac{1}{4}\begin{pmatrix} 3 \\ 1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$Y_3 = A^3 Y_0 = \frac{1}{4}(1)^3 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \frac{1}{4}(0.2)^3 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 3.008 \\ 0.992 \end{pmatrix}$$

Probability in good mood in day 3 $= \dfrac{3.008}{4}$.

(c) $$A^n Y_0 = \frac{1}{4}(1)^n \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \underbrace{\frac{1}{4}(0.2)^n \begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{\to 0 \text{ as } n \to \infty}$$

$$\xrightarrow{n \to \infty} \frac{1}{4}\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

In the long run, probability in good mood $= \dfrac{3}{4}$ (75%)

#5.21 $A = \begin{bmatrix} 0.7 & 0.2 & 0.5 \\ 0.1 & 0.6 & 0.3 \\ 0.2 & 0.2 & 0.2 \end{bmatrix}$

(a) $V_0 = \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ (Monday)

$$V_1 = AV_0 = \frac{1}{3}\begin{bmatrix} 0.7 & 0.2 & 0.5 \\ 0.1 & 0.6 & 0.3 \\ 0.2 & 0.2 & 0.2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1.4 \\ 1 \\ 0.6 \end{bmatrix} \quad \text{(Tuesday)}$$

$$V_2 = AV_1 = \frac{1}{3}\begin{bmatrix} 0.7 & 0.2 & 0.5 \\ 0.1 & 0.6 & 0.3 \\ 0.2 & 0.2 & 0.2 \end{bmatrix}\begin{bmatrix} 1.4 \\ 1 \\ 0.6 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1.48 \\ 0.92 \\ 0.60 \end{bmatrix} \quad \text{(Wednesday)}$$

1b) $(A - I \mid \vec{0}) \rightarrow \left(\begin{array}{ccc|c} -0.3 & 0.2 & 0.5 & 0 \\ 0.1 & -0.4 & 0.3 & 0 \\ 0.2 & 0.2 & -0.8 & 0 \end{array}\right)$

$\rightarrow \left(\begin{array}{ccc|c} 1 & -4 & 3 & 0 \\ -3 & 2 & 5 & 0 \\ 2 & 2 & -8 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & -4 & 3 & 0 \\ 0 & -10 & 14 & 0 \\ 0 & 10 & -14 & 0 \end{array}\right)$

$\rightarrow \left(\begin{array}{ccc|c} 1 & -4 & 3 & 0 \\ 0 & 1 & -7/5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -13/5 & 0 \\ 0 & 1 & -7/5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$X_1 = \begin{pmatrix} \frac{13}{5}\alpha \\ \frac{7}{5}\alpha \\ \alpha \end{pmatrix} = \begin{pmatrix} 13/25 \\ 7/25 \\ 5/25 \end{pmatrix} \quad \left(\alpha = \frac{5}{25}\right)$

(c) $\lambda = 0$: $(A - 0I \mid \vec{0}) \rightarrow \left(\begin{array}{ccc|c} 0.7 & 0.2 & 0.5 & 0 \\ 0.1 & 0.6 & 0.3 & 0 \\ 0.2 & 0.2 & 0.2 & 0 \end{array}\right)$

$\rightarrow \left(\begin{array}{ccc|c} 1 & 6 & 3 & 0 \\ 7 & 2 & 5 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 6 & 3 & 0 \\ 0 & -40 & -16 & 0 \\ 0 & -5 & -2 & 0 \end{array}\right)$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 6 & 3 & 0 \\ 0 & 1 & 2/5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 3/5 & 0 \\ 0 & 1 & 2/5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$X_2 = \begin{pmatrix} -3 \\ -2 \\ 5 \end{pmatrix}$$

$$\lambda = 0.5, \quad (A - 0.5I) \rightarrow \left(\begin{array}{ccc|c} 0.2 & 0.2 & 0.5 & 0 \\ 0.1 & 0.1 & 0.3 & 0 \\ 0.2 & 0.2 & -0.3 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 2 & 2 & 5 & 0 \\ 2 & 2 & -3 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -9 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$X_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

(d) $V_0 = \frac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = c_1\begin{bmatrix}13/25\\7/25\\5/25\end{bmatrix} + c_2\begin{bmatrix}-3\\-2\\5\end{bmatrix} + c_3\begin{bmatrix}-1\\1\\0\end{bmatrix}$

$$A^n V_0 = c_1(1)^n\begin{bmatrix}13/25\\7/25\\5/25\end{bmatrix} + c_2(0)^n\begin{bmatrix}-3\\-2\\5\end{bmatrix} + c_3(0.5)^n\begin{bmatrix}-1\\1\\0\end{bmatrix}$$

$$= c_1\begin{bmatrix}13/25\\7/25\\5/25\end{bmatrix} + c_3(0.5)^n\begin{bmatrix}-1\\1\\0\end{bmatrix}$$

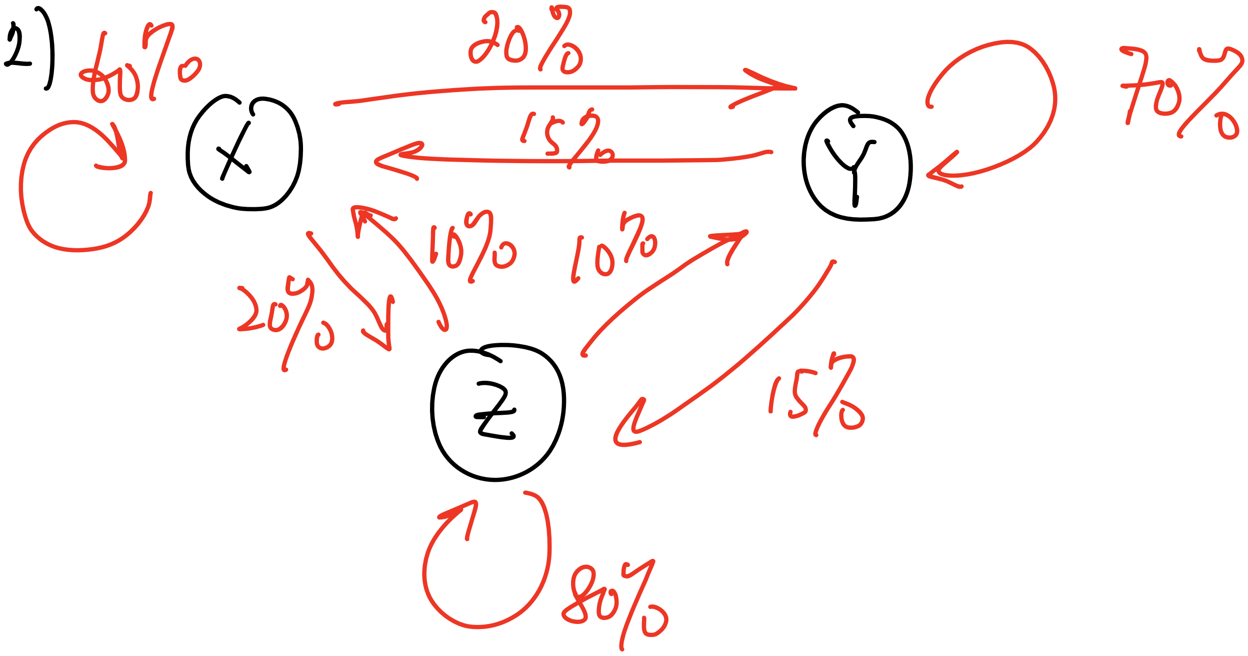
(e) For any V_0, write V_0 as $c_1X_1 + c_2X_2 + c_3X_3$.

Then $A^nV_0 = c_1(1)^nX_1 + c_2(0)^nX_2 + c_3(0.5)^nX_3$

$$= c_1X_1 + c_3(0.5)^nX_3$$

$$\to c_1X_1 \text{ as } n \to \infty.$$

(5.22)



(a) $A = \begin{bmatrix} 0.6 & 0.15 & 0.1 \\ 0.2 & 0.7 & 0.1 \\ 0.2 & 0.15 & 0.8 \end{bmatrix}$

(b) $(A - I \mid 0) \rightarrow \left[\begin{array}{ccc|c} -0.4 & 0.15 & 0.1 & 0 \\ 0.2 & -0.3 & 0.1 & 0 \\ 0.2 & 0.15 & -0.2 & 0 \end{array}\right]$

$\rightarrow \left[\begin{array}{ccc|c} -4 & 1.5 & 1 & 0 \\ 2 & -3 & 1 & 0 \\ 2 & 1.5 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 2 & -3 & 1 & 0 \\ -4 & 1.5 & 1 & 0 \\ 2 & 1.5 & -2 & 0 \end{array}\right]$

$\rightarrow \left[\begin{array}{ccc|c} 2 & -3 & 1 & 0 \\ 0 & -4.5 & 3 & 0 \\ \cancel{0} & \cancel{4.5} & \cancel{-3} & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 2 & -3 & 1 & 0 \\ 0 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$$\to \left[\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 0 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1/2 & 0 \\ 0 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{\alpha}{2} \\ \frac{2\alpha}{3} \\ \alpha \end{pmatrix} \xrightarrow{\alpha = 50} \begin{pmatrix} \frac{300}{13} \\ \frac{400}{13} \\ \frac{600}{13} \end{pmatrix} (\%)$$

Choose α s.t. $\frac{\alpha}{2} + \frac{2\alpha}{3} + \alpha = 100\ (\%)$

$$\alpha = \frac{100}{\frac{1}{2} + \frac{2}{3} + 1} = \frac{600}{13}$$

(check:

$$\begin{bmatrix} 0.6 & 0.15 & 0.1 \\ 0.2 & 0.7 & 0.1 \\ 0.2 & 0.15 & 0.8 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

)

p.290 #5.29

(b) $A = \begin{pmatrix} 1 & 2 & 0 \\ -3 & 2 & 3 \\ -1 & 2 & 2 \end{pmatrix}$ $p(\lambda) = (\lambda-1)(\lambda-2)^2$

$\lambda=2$: $(A-2I) \rightarrow \begin{pmatrix} -1 & 2 & 0 \\ -3 & 0 & 3 \\ -1 & 2 & 0 \end{pmatrix}$

$\rightarrow \begin{pmatrix} -1 & 2 & 0 \\ 1 & 0 & -1 \\ -1 & 2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ Only 1 free var.

Hence $\lambda=2$ is defective.

Hence A is not diagonalizable.

5.31 $p(\lambda) = \det(A - \lambda I)$

$$= (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$$

distinct λ_i

Hence there are n lin. ind. eigenvectors.

Hence A is diagonalizable:

$$A = QDQ^{-1}$$

$$\det A = \det(QDQ^{-1})$$

$$= (\det Q)(\det D)(\det Q)^{-1}$$

$$= \det D = \det \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

$$= \lambda_1 \lambda_2 \cdots \lambda_n.$$

5.32 : $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix} = QDQ^{-1}$

$$= Q \begin{bmatrix} 1 & & \\ & 1 & \\ & & 3 \end{bmatrix} Q^{-1}$$

$$= Q \begin{bmatrix} \sqrt{1} & & \\ & \sqrt{1} & \\ & & \sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{1} & & \\ & \sqrt{1} & \\ & & \sqrt{3} \end{bmatrix} Q^{-1}$$

$$= \underbrace{Q \begin{bmatrix} 1 & & \\ & 1 & \\ & & \sqrt{3} \end{bmatrix} Q^{-1}}_{B} \underbrace{Q \begin{bmatrix} 1 & & \\ & 1 & \\ & & \sqrt{3} \end{bmatrix} Q^{-1}}_{B}$$

$$= B^2, \quad \left(\text{Set } B = Q \begin{bmatrix} 1 & & \\ & 1 & \\ & & \sqrt{3} \end{bmatrix} Q^{-1} \right)$$

5.33 $A = QDQ^{-1}$; $D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$

$\lambda_i = \pm 1$

$$A^2 = QDQ^{-1}QDQ^{-1}$$

$$= QD^2Q^{-1} = Q\underbrace{\begin{bmatrix} \lambda_1^2 & & & \\ & \lambda_2^2 & & \\ & & \ddots & \\ & & & \lambda_n^2 \end{bmatrix}}_{I}Q^{-1} \quad (\lambda_i^2 = 1)$$

$$= QIQ^{-1} = \underline{I}$$

5.34 $A = QDQ^{-1}$; $D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

$\underline{\lambda_i = 0 \text{ or } 1}$ $(\lambda_i^2 = \lambda_i)$

$$A^2 = QD^2Q^{-1}$$

$$= Q\begin{bmatrix} \lambda_1^2 & & \\ & \ddots & \\ & & \lambda_n^2 \end{bmatrix}Q^{-1} = Q\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}Q^{-1}$$

$$= \underline{QDQ^{-1} = A}$$

5.35. $A = QDQ^{-1}$

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

λ_i: 2 or 4.

$$A^2 = QD^2Q^{-1} = Q\begin{bmatrix} \lambda_1^2 & & \\ & \ddots & \\ & & \lambda_n^2 \end{bmatrix}Q^{-1}$$

$I = QQ^{-1}$

$$A^2 - 6A + 8I = QD^2Q^{-1} - 6QDQ^{-1} + 8I$$

$$= Q\underbrace{[D^2 - 6D + 8I]}_{\begin{bmatrix} \ddots & & \\ & \boxed{\lambda_i^2 - 6\lambda_i + 8} = 0 & \\ & & \ddots \end{bmatrix} = 0}Q^{-1} = 0$$

as $\lambda_i = 2$ or 4

5.36 $\quad A = QDQ^{-1}$

$$A^k = QD^kQ^{-1}$$

$$a_nA^n + a_{n-1}A^{n-1} + \cdots + a_1A + a_0I$$

$$= a_nQD^nQ^{-1} + a_{n-1}QD^{n-1}Q^{-1} + \cdots + a_1QDQ^{-1} + a_0QQ^{-1}$$

$$= Q\Big[\underbrace{a_nD^n + a_{n-1}D^{n-2} + \cdots + a_1D + a_0I}\Big]Q^{-1}$$

$$\underbrace{\begin{bmatrix} \ddots & & \\ & \underbrace{\boxed{a_n\lambda_i^n + a_{n-1}\lambda_i^{n-1} + \cdots + a_1\lambda_i + a_0}}_{=q(\lambda_i)=0} & \\ & & \ddots \end{bmatrix}}_{=O}$$

$$= O$$

5.37 $\quad A = \begin{bmatrix} 2 & a & b \\ 0 & -5 & c \\ 0 & 0 & 2 \end{bmatrix}$

$\lambda = 2, 2, -5$

$\left(\det(A - \lambda I) = \det \begin{pmatrix} 2-\lambda & a & b \\ 0 & -5-\lambda & c \\ 0 & 0 & 2-\lambda \end{pmatrix} = (2-\lambda)^2(-5-\lambda) \right)$

$\lambda = 5$ (no problem: $1 \leq g \leq m = 1 \Rightarrow 1 = g = m = 1$)

$\underline{\lambda = 2}$: $(A - 2I)X = 0 \rightarrow \left(\begin{array}{ccc|c} 0 & a & b & 0 \\ 0 & -7 & c & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$(m=2)$ (Need 2 free vars.)

$\rightarrow \left(\begin{array}{ccc|c} 0 & 1 & -c/7 & 0 \\ 0 & a & b & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 0 & 1 & -c/7 & 0 \\ 0 & 0 & b+\frac{ac}{7} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

free $\qquad$ free if $b + \frac{ac}{7} = 0$

i.e. $\boxed{7b + ac = 0}$

5.38 (a) $A = \begin{bmatrix} 2 & 1 & a & b \\ 0 & 3 & -1 & c \\ 0 & 0 & 2 & d \\ 0 & 0 & 0 & 3 \end{bmatrix}$ $\lambda = 2, 2, 3, 3$ (why?)

$\lambda = 2 \Rightarrow (A - 2I \mid 0) \rightarrow \left(\begin{array}{cccc|c} 0 & 1 & a & b & 0 \\ 0 & 1 & -1 & c & 0 \\ 0 & 0 & 0 & d & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$

Need 2 free vars.

$\rightarrow \left(\begin{array}{cccc|c} 0 & 1 & a & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 0 & 1 & a & 0 & 0 \\ 0 & 0 & -1-a & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

$-1-a=0$ i.e. $a=-1$

$\lambda = 3 \Rightarrow (A - 3I \mid 0) \rightarrow \left(\begin{array}{cccc|c} +1 & -1 & -a & -b & 0 \\ 0 & 0 & +1 & -c & 0 \\ 0 & 0 & +1 & -d & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

Need 2 free vars.

$\rightarrow \left(\begin{array}{cccc|c} 1 & -1 & -a & -b & 0 \\ 0 & 0 & 1 & -c & 0 \\ 0 & 0 & 0 & c-d & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

$c-d=0$

i.e. $c=d$

5.38 (b) $A = \begin{bmatrix} 2 & -2 & a & b \\ 0 & 1 & c & d \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ $\lambda = 1, 2, 2, 2$

$\lambda = 1$ (no problem)

$\lambda = 2$ (Need 3 free vars)

$$(A - 2I \mid 0) \rightarrow \left(\begin{array}{cccc|c} 0 & -2 & a & b & 0 \\ 0 & -1 & c & d & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|c} 0 & 1 & -c & -d & 0 \\ 0 & -2 & a & b & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|c} 0 & 1 & -c & -d & 0 \\ 0 & 0 & \underline{a-2c} & \underline{b-2d} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Need $a = 2c$, $b = 2d$

Additional Problem #1

(a) $(*)\quad c_1 X_1 + c_2 X_2 + c_3 X_3 + c_4 Y_1 + c_5 Y_2 + c_6 Y_3 = 0$

$\boxed{c_1, c_2, c_3, c_4, c_5, c_6 \text{ not all } 0.}$

$\lambda_2 (*) \Rightarrow$

$$c_1 \lambda_2 X_1 + c_2 \lambda_2 X_2 + c_3 \lambda_2 X_3 + c_4 \lambda_2 Y_1 + c_5 \lambda_2 Y_2 + c_6 \lambda_2 Y_3 = 0$$

$A (*) \Rightarrow$

$$c_1 \lambda_1 X_1 + c_2 \lambda_1 X_2 + c_3 \lambda_1 X_3 + c_4 \lambda_2 Y_1 + c_5 \lambda_2 Y_2 + c_6 \lambda_2 Y_3 = 0$$

$A(*) - \lambda_2 (*) \Rightarrow$

$$\underbrace{c_1 (\lambda_1 - \lambda_2)}_{=0} X_1 + \underbrace{c_2 (\lambda_1 - \lambda_2)}_{=0} X_2 + \underbrace{c_3 (\lambda_1 - \lambda_2)}_{=0} X_3 = 0$$

Since X_1, X_2, X_3 – lin ind

$\lambda_1 \neq \lambda_2 \Rightarrow \quad c_1 = 0, \ c_2 = 0, \ c_3 = 0$

Back to $(*) \Rightarrow \quad c_4 Y_1 + c_5 Y_2 + c_6 Y_3 = 0$

$\Rightarrow \quad c_4 = 0, \ c_5 = 0, \ c_6 = 0$

Since Y_1, Y_2, Y_3 – lin ind

$\boxed{\text{So all of } c_1, c_2, c_3, c_4, c_5, c_6 \text{ are zero ?!!}}$

Additional Problem #2

(a) (i) $A = IAI^{-1}$

(ii) $A = PBP^{-1} \Longrightarrow B = P^{-1}AP$

$$= (P^{-1})A(P^{-1})^{-1}$$

(iii) $A = PBP^{-1}$, $B = QCQ^{-1}$

$$= P(QCQ^{-1})P^{-1}$$

$$= (PQ)C\,Q^{-1}P^{-1}$$

$$= (PQ)\,C\,(PQ)^{-1} \qquad \left((PQ)^{-1} = Q^{-1}P^{-1}\right)$$

(b) $\det(A) = \det(PBP^{-1}) = (\cancel{\det P})(\det B)\underbrace{(\det P^{-1})}_{\cancel{(\det P)^{-1}}}$

$$= (\det B)$$

(c) $\operatorname{tr} A = \operatorname{tr}(PBP^{-1}) = \operatorname{tr}(\cancel{P^{-1}}\cancel{P}B) = \operatorname{tr} B$

(d) $\det(A - \lambda I) = \det(PBP^{-1} - \lambda PP^{-1})$

$$= \det\left(P(B - \lambda I)P^{-1}\right)$$

$$= (\cancel{\det P})\det(B - \lambda I)(\cancel{\det P})^{-1}$$

$$= \det(B - \lambda I)$$

(e) $AX = \lambda X \iff PBP^{-1}X = \lambda X$

$$\iff B\underbrace{(P^{-1}X)}_{Y} = \lambda \underbrace{(P^{-1}X)}_{Y}$$

i.e. $BY = \lambda Y$

Hence $\boxed{\begin{array}{c} AX = \lambda X \iff BY = \lambda Y \\ Y = P^{-1}X \text{ or } X = PY \end{array}}$