MA 351 Fall 2024 Selected Hw Solns.

Hw5 p. 92

#1.116 $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in W$, but $(-1)\begin{pmatrix} 1 \\ 1 \end{pmatrix} \notin W$

#1.117 $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in W$, but $\frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \notin W$

#1.118 $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$: $x^2 = y+z$

$\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 1 \\ 8 \end{pmatrix}$ satisfies $x^2 = y+z$

But $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \\ 8 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 11 \end{pmatrix}$ does not satisfies $x^2 = y+z$

(S, T are subspaces in 1.131, 1.132, 1.133)

intersection, AND

#1.131 Consider $S \cap T$. Let $u, v \in S \cap T$.

$$u \in S,\ v \in S \qquad\qquad u \in T,\ v \in T$$
$$\Longrightarrow \alpha u + \beta v \in S \qquad\qquad \Longrightarrow \alpha u + \beta v \in T$$

Hence $\alpha u + \beta v \in S \cap T$.

$\Longrightarrow$ $S \cap T$ is a subspace.

#1.132. For example

$$S = \left\{\begin{pmatrix} x \\ 0 \end{pmatrix}\right\} - x\text{-axis}, \quad T = \left\{\begin{pmatrix} 0 \\ y \end{pmatrix}\right\} - y\text{-axis}$$

union, OR

$$S \cup T = \left\{\begin{pmatrix} x \\ 0 \end{pmatrix}\right\} \cup \left\{\begin{pmatrix} 0 \\ y \end{pmatrix}\right\} = \left\{\begin{pmatrix} x \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ y \end{pmatrix}\right\}.$$

Consider $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in S \cup T$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in S \cup T$

But $\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \notin S \cup T$

Hence $S \cup T$ is not a subspace

In order for $S \cup T$ to be a subspace:

say $u \in S$, $v \in T$, we need

$$\underbrace{u+v \in S}_{\Downarrow} \text{ or } \underbrace{u+v \in T}_{\Downarrow}$$

Since $u \in S \Rightarrow v \in S$ Since $v \in T \Rightarrow u \in T$

Hence if $T \subseteq S$ or $S \subseteq T$

i.e. if one subspace is completely contained in another, then $S \cup T = S$ or $S \cup T = T$.

In this case $S \cup T$ will be a subspace

#1.133

$$S+T = \{ u+v : u \in S \text{ and } v \in T \}$$

Let $p = u+v \in S+T$, $u \in S$, $v \in T$

$q = w+z \in S+T$ where $w \in S$, $z \in T$

$$\alpha p + \beta q = \alpha(u+v) + \beta(w+z)$$
$$= \underbrace{(\alpha u + \beta w)}_{\in S} + \underbrace{(\alpha v + \beta z)}_{\in T} \in S+T$$

Hence $S+T$ is a subspace.

Hw 6 #2.19

$$Z_1 = a_1X + b_1Y, \quad Z_2 = a_2X + b_2Y, \quad Z_3 = a_3X + b_3Y$$

$$c_1Z_1 + c_2Z_2 + c_3Z_3 = 0$$

$$c_1(a_1X + b_1Y) + c_2(a_2X + b_2Y) + c_3(a_3X + b_3Y) = 0$$

$$\underbrace{(a_1c_1 + a_2c_2 + a_3c_3)}_{\text{Set to } 0}X + \underbrace{(b_1c_1 + b_2c_2 + b_3c_3)}_{\text{Set to } 0}Y = 0$$

$$\begin{aligned} a_1c_1 + a_2c_2 + a_3c_3 &= 0 \\ b_1c_1 + b_2c_2 + b_3c_3 &= 0 \end{aligned}$$

2 eqns in 3 unknowns (c_1, c_2, c_3)

There must be free variables and hence non-trivial solutions.

$\Rightarrow$ Z_1, Z_2, Z_3 must be lin dep.

Hw6

p. 123 # 2.32(d)

$$W = \left\{ \begin{pmatrix} a+2b-4c+5d \\ -2a-2b+2c-6d \\ 6a+4b+14d \\ 3a+b+3c+5d \end{pmatrix}, \begin{matrix} a,b,c,d \\ \in \mathbb{R} \end{matrix} \right\}$$

$$= \left\{ a\begin{pmatrix} 1 \\ -2 \\ 6 \\ 3 \end{pmatrix} + b\begin{pmatrix} 2 \\ -2 \\ 4 \\ 1 \end{pmatrix} + c\begin{pmatrix} -4 \\ 2 \\ 0 \\ 3 \end{pmatrix} + d\begin{pmatrix} 5 \\ -6 \\ 14 \\ 5 \end{pmatrix} \right\}$$

$$= \text{Span}\left\{ \underbrace{\begin{pmatrix} 1 \\ -2 \\ 6 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} -4 \\ 2 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 5 \\ -6 \\ 14 \\ 5 \end{pmatrix}}_{} \right\}$$

(determine which is redundent)

$$\begin{pmatrix} 1 & 2 & -4 & 5 \\ -2 & -2 & 2 & -6 \\ 6 & 4 & 0 & 14 \\ 3 & 1 & 3 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -4 & 5 \\ 0 & 2 & -6 & 4 \\ 0 & -8 & 24 & -16 \\ 0 & -5 & 15 & -10 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & -4 & 5 \\ 0 & 1 & -3 & 2 \\ 0 & 1 & -3 & 2 \\ 0 & 1 & -3 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 4 & 5 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

redundant

Hence $$W = \text{Span}\Big\{ \underbrace{\begin{pmatrix} 1 \\ -2 \\ 6 \\ 1 \end{pmatrix}}_{u_1}, \underbrace{\begin{pmatrix} 2 \\ -2 \\ 4 \\ 1 \end{pmatrix}}_{u_2} \Big\}$$

$$\underbrace{u_1 \quad u_2}_{\text{Basis vectors}}$$

$$\dim W = 2$$

Other basis vectors:

$$W = \text{Span}\{2u_1, 2u_2\}$$

$$= \text{Span}\{u_1 + u_2, u_1 - u_2\}$$

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p.123 #2.33 (b)

$$W = \left\{ \begin{bmatrix} a+b+2c+3d & a+b+2c+3d \\ 2a+2c+4d & a+b+2c+3d \end{bmatrix},\ a,b,c,d \in \mathbb{R} \right\}$$

$$= \left\{ a\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} + b\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + c\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} + d\begin{pmatrix} 3 & 3 \\ 4 & 3 \end{pmatrix},\ a,b,c,d \in \mathbb{R} \right\}$$

$$= \operatorname{Span}\left\{ \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 3 \\ 4 & 3 \end{bmatrix} \right\}$$

(Check for redundancy)

$$\begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 1 & 2 & 3 \\ 2 & 0 & 2 & 4 \\ 1 & 1 & 2 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \underbrace{0 \quad\quad 0}_{\text{redundant.}} \end{pmatrix}$$

Hence $W = \text{Span}\Big\{ \underbrace{\underset{u_1}{\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}}, \underset{u_2}{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}}_{\text{Basis vectors}} \Big\}$

$$\dim W = 2$$

Other possible basis:

$$W = \text{Span}\{2u_1, -u_2\}$$
$$= \text{Span}\{u_1 + 2u_2,\ u_1 - u_2\}$$

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#2.35 (a) 2×2 $A = A^t$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \Rightarrow \begin{matrix} \cancel{a=a}, & b=c \\ c=b & \cancel{d=d} \end{matrix}$$

$$A = \begin{pmatrix} a & b \\ b & d \end{pmatrix} \qquad a, b, d - \text{free}$$

$$= a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\boxed{\text{Basis} = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}}$$

why lin ind?

(c) 2×2, $A = -A^t$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = -\begin{pmatrix} a & c \\ b & d \end{pmatrix} \Rightarrow \begin{matrix} a=-a \Rightarrow a=0 \\ b=-c, \quad c=-b \\ d=-d \Rightarrow d=0 \end{matrix}$$

$$A = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} = b\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad b - \text{free}$$

$$\text{Basis} = \left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}$$

p. 125, #2.46

Suppose $\{A, B\}$ is a basis for $W = \text{Span}\{A, B\}$

$$X = 2A + 3B, \quad Y = 3A - 5B$$

Show that $\{X, Y\}$ is also a basis for W

It suffices to check for lin. ind. of X & Y

(Thm 2.8)

$$c_1 X + c_2 Y = 0$$

$$c_1(2A + 3B) + c_2(3A - 5B) = 0$$

$$\underbrace{(2c_1 + 3c_2)}_{=0} A + \underbrace{(3c_1 - 5c_2)}_{=0} B = 0$$

Hence
$$\left.\begin{array}{l} 2c_1 + 3c_2 = 0 \\ 3c_1 - 5c_2 = 0 \end{array}\right\} \Longrightarrow \begin{array}{l} c_1 = 0 \\ c_2 = 0 \end{array}$$

So that $\{X, Y\}$ is lin ind.

(Thm 2.8)

(Though not necessary, just to be sure that X, Y can represent any lin comb of A, B:

Given $aA + bB$, find c_1, c_2 s.t.

$$c_1 X + c_2 Y = aA + bB$$

$$c_1(2A + 3B) + c_2(3A - 5B) = aA + bB$$

$$\begin{cases} 2c_1 + 3c_2 = a \\ 3c_1 - 5c_2 = b \end{cases} \Rightarrow \left(\begin{array}{cc|c} 2 & 3 & a \\ 3 & -5 & b \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{cc|c} 1 & 3/2 & a/2 \\ 3 & -5 & b \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 1 & 3/2 & a/2 \\ 0 & -\frac{19}{2} & b - \frac{3a}{2} \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{cc|c} 1 & 3/2 & a/2 \\ 1 & 1 & \frac{3}{19}a - \frac{2}{19}b \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 0 & \frac{5}{19}a + \frac{3}{19}b \\ 0 & 1 & \frac{3}{19}a - \frac{2}{19}b \end{array}\right)$$

$$\boxed{\underbrace{\left(\frac{5}{19}a + \frac{3}{19}b\right)}_{c_1} X + \underbrace{\left(\frac{3}{19}a - \frac{2}{19}\right)}_{c_2} Y = aA + bB}$$

Hw 7

#2.65 $A = \begin{pmatrix} 2 & 1 & 3 & 1 \\ 1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 1 \\ 3 & 3 & 8 & 2 \end{pmatrix}$

(a)

$$\to \begin{pmatrix} 1 & 1 & 3 & 0 \\ 2 & 1 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 3 & 3 & 8 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 3 & 0 \\ 0 & -1 & -3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & -1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & 0 & 6 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} = R$$

(b) Basis for Row(A) $= \left\{ \begin{array}{ll} (1\ \ 0\ \ 0\ \ 1), & \leftarrow r_1 \\ (0\ \ 1\ \ 0\ \ 5), & \leftarrow r_2 \\ (0\ \ 0\ \ 1\ \ -2) & \leftarrow r_3 \end{array} \right\}$

(c) $(2 \quad 1 \quad 3 \quad 1) = 2(1 \quad 0 \quad 0 \quad 1)$
$+1(0 \quad 1 \quad 0 \quad 5)$
$+3(0 \quad 0 \quad 1 \quad -2)$

$$\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ 2 & 1 & 3 & 1 \end{array}$$

(d) Col's of R cannot span Col(A)

Because last entries of col's of R are all zeros.

2.74 Each row of the table indicated in Figure 2.3 summarizes data collected for some matrix A. The "always solvable" column refers to whether the system $AX = B$ is solvable for all B and the "unique solution" column refers to whether or not there is at most one solution. Your assignment is to fill in all the missing data from the table. Where the data is inconsistent, write "impossible."

Size	*Always Solvable*	*Unique Solution*	*Dimension of Column Space*	*Dimension of Nullspace*	*Rank*
4×3	yes	Impossible			
3×4	yes	No	3	1	3
5×5	No	No	4	1	4
5×5	Yes	Yes	5	0	5
3×2	No	yes	2	0	2
4×4	yes	Yes	4	0	4
5×4	No	No	3	1	3
5×4		Impossible			5

Homework 8 — p.145 #2.76

$$A = \begin{bmatrix} 1 & 1 & 3 & 2 \\ 2 & 2 & 6 & 4 \\ 10 & 2 & 14 & 20 \\ 2\sqrt{2} & -\sqrt{2} & 0 & 4\sqrt{2} \\ \pi & e & \pi+2e & 2\pi \\ \sqrt{2} & \sqrt{3} & \sqrt{2}+2\sqrt{3} & 2\sqrt{2} \\ \ln 5 & 6 & \ln 5+12 & 2\ln 5 \\ -7 & 4 & 1 & -14 \\ 17 & -24 & -31 & 34 \\ 2 & 2 & 6 & 4 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & -8 & -16 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & e-\pi & 2e-2\pi & 0 \\ 0 & \sqrt{3}-\sqrt{2} & 2\sqrt{3}-2\sqrt{2} & 0 \\ 0 & 6-\ln 5 & 12-2\ln 5 & 0 \\ 0 & 11 & 22 & 0 \\ 0 & -41 & -82 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 3 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\cdots$$

(a) $\text{Rank}(A) = 2$ (= # of pivots)

(b) $\text{Nullity}(A) = 2$ (= # of free vars.)

(c) $\text{Rank}(A) = 2 < 10$

(So $AX = B$ not solvable for all B.)

(d) If $AX = B$ is solvable, it has infinitely many soln. (due to free var.)

(e) Basis for Row (A)

$\{(1 \quad 1 \quad 3 \quad 2),\ (0 \quad 1 \quad 2 \quad 0)\}$ ← REF

or

$\{(1 \quad 0 \quad 1 \quad 2),\ (0 \quad 1 \quad 2 \quad 0)\}$ ← RREF

(f) $\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$ ← RREF of A

$$\begin{bmatrix} 1 & 1 & 3 & 2 \\ -7 & 4 & 1 & -14 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & 3 & 2 \\ 0 & 11 & 22 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & 3 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

The same RREF as A.

p. 146 #2.78

$$A = \begin{bmatrix} 1 & 2 & e & \ln 4 & 7 & \pi \\ 1 & 2 & \pi & \sqrt{2} & 1 & \pi \\ 2 & 4 & e+\pi & \sqrt{2}+\ln 4 & 8 & 2\pi \\ 3 & 6 & 2e+\pi & 2\ln 4+\sqrt{2} & 15 & 3\pi \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & e & \ln 4 & 7 & \pi \\ 0 & 0 & \pi-e & \sqrt{2}-\ln 4 & -6 & 0 \\ 0 & 0 & \pi-e & \sqrt{2}-\ln 4 & -6 & 0 \\ 0 & 0 & \pi-e & \sqrt{2}-\ln 4 & -6 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & e & \ln 4 & 7 & \pi \\ 0 & 0 & \pi-e & \sqrt{2}-\ln 4 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) $\text{Rank}(A) = 2$

(b) $\begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} e \\ \pi \\ e+\pi \\ 2e+\pi \end{pmatrix}$

(u_1) (u_2)

Another basis: u_1+u_2, u_1-u_2 (e.g)

(c) Check: $A \begin{pmatrix} -\pi \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \tilde{0}$

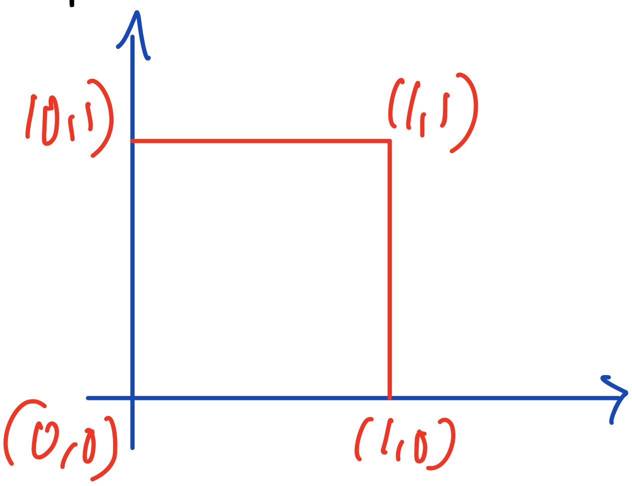
$A \begin{pmatrix} -2 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \tilde{0}$

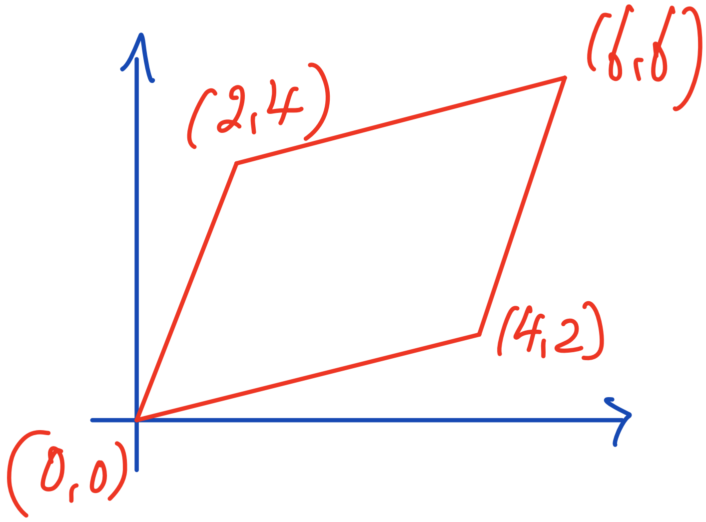
They do not ~~span~~ span $Null(A)$ as $\dim(Null(A)) = 4$.

(d) $A \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix}$

(e) $A \begin{pmatrix} \alpha \\ \beta \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix}$ for $\alpha + 2\beta = 1$

p. 158 #3.5, #3.6





#3.5

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \Longrightarrow \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \Longrightarrow \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

(Automatically, $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$ why?)

#3.6

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{aligned} 4p + 2q &= 1 \\ 4r + 2s &= 0 \end{aligned}$$

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \begin{aligned} 2p + 4q &= 0 \\ 2r + 4s &= 1 \end{aligned}$$

$$\begin{bmatrix} 1/3 & -1/6 \\ -1/6 & 1/3 \end{bmatrix}$$

Note: $\boxed{\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1/3 & -1/6 \\ -1/6 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$

#3.10 $X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \longrightarrow Y_1 = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

$X_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2X_1 \longrightarrow 2Y_1 = \begin{pmatrix} 8 \\ 10 \end{pmatrix} \neq \begin{pmatrix} 5 \\ 6 \end{pmatrix}$

Hence it is not possible to find A.

#3.11 $X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \longrightarrow Y_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$X_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \longrightarrow Y_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$X_3 = \begin{pmatrix} 3 \\ 4 \end{pmatrix} = X_1 + X_2 \longrightarrow Y_1 + Y_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

Hence it is not possible to find A.

p. 160 # 3.15

(a) $T\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2x+3y-7z\\0\end{pmatrix} = \begin{pmatrix}2 & 3 & -7\\0 & 0 & 0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$

(b) $T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x+y\\x-y\\x+2y\end{pmatrix} = \begin{pmatrix}1 & 1\\1 & -1\\1 & 2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$

(c) $T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2x+3y\\2y+x\\-3x-y\end{pmatrix} = \begin{pmatrix}2 & 3\\1 & 2\\-3 & -1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$

(d) $T\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix} = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0 & 0\\0 & 1 & 0 & 0 & 0\\0 & 0 & 1 & 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}$

(e) $T\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2x-3y-7z\\y-z\end{pmatrix} = \begin{pmatrix}2 & -3 & -7\\0 & 1 & -1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$

(f) $T\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}z\\y\\x\end{pmatrix} = \begin{pmatrix}0 & 0 & 1\\0 & 1 & 0\\1 & 0 & 0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$

p. 176, #3.44

$$\overset{A}{\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}} \overset{B}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \overset{=}{=} \overset{B}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \overset{A}{\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}}$$

$$\begin{bmatrix} a+2c & b+2d \\ 3c & 3d \end{bmatrix} = \begin{bmatrix} a & 2a+3b \\ c & 2c+3d \end{bmatrix}$$

$a+2c=a \Longrightarrow c=0$

$b+2d=2a+3b \Longrightarrow 2a+2b-2d=0$

$3c=c \Longrightarrow c=0$

$3d=2c+3d \Longrightarrow c=0$

all we need is $a+b-d=0$

b, d – free,

$a=-b+d$

Hence

$$B=\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -b+d & b \\ 0 & d \end{pmatrix} = b\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} + d\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(Basis vector)

Homework 9 p. 175 # 3.41

$$\underbrace{\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}}_{B} = \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}_{O}$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ c \\ e \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \to \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$a = -e$

$c = 0$

$$\Rightarrow \begin{pmatrix} a \\ c \\ e \end{pmatrix} = e \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Similarly,

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} b \\ d \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} b \\ d \\ f \end{pmatrix} = f \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Hence

$$B = \begin{pmatrix} -e & -f \\ 0 & 0 \\ e & f \end{pmatrix}$$ which has rank $= 1$.

p.175 #3.42

$$C = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ a_{31} & a_{23} & \cdots \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 3 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\to \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} = \begin{pmatrix} -a_{31} \\ 0 \\ a_{31} \end{pmatrix} = a_{31} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Hence, all the columns of B are parallel to $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

So $\text{Rank}(B) = 1$.

Homework 9

p.190 #3.72

$$A \quad \left[\begin{array}{ccc|ccc} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & a & 0 & 1 & 0 & -b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$-b+ac$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -a & \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & -a & -b+ac \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$$

p. 192 #3.74

(a) $N^3 = I$

(b) Observe: $(I-N)(I+N+N^2)$

$$= \begin{array}{l} I + \cancel{N} + \cancel{N^2} \\ -\cancel{N} - \cancel{N^2} - \cancel{N^3} \end{array} = I$$

0

Hence $\boxed{(I-N)^{-1} = I+N+N^2}$

(Note: for square matrix A,
if $AB = I$, then $BA = I$
& hence $B = A^{-1}$)

(c) Suppose $N^4 = I$

Then similarly, $(I-N)(I+N+N^2+N^3) = I$

Hence $\boxed{(I-N)^{-1} = I+N+N^2+N^3}$

#3.75

$$XA = C$$

$$X\cancel{AB} = CB \Rightarrow \underline{X = CB}$$

#3.76

$$AY = C$$

$$\cancel{BA}\,Y = BC \Rightarrow \underline{Y = BC}$$

#3.77

$$A^2 + 3A + I = 0$$

$$A^2 + 3A = -I$$

$$A(A^2 + 3A) = -I$$

$$A\underbrace{(-A^2 - 3A)} = I$$

$$A^{-1} = -A^2 - 3A$$

#3.78

$$A^3 + 3A^2 + 2A + 5I = 0$$

$$A^3 + 3A^2 + 2A = -5I$$

$$A\underbrace{\left(-\frac{A^2}{5} - \frac{3A}{5} - \frac{2I}{5}\right)} = I$$

$$A^{-1} = -\frac{A^2}{5} - \frac{3A}{5} - \frac{2I}{5}$$