MA 351: Introduction to Linear Algebra and Its Applications
Fall 2024, Final Exam
Instructor: Yip

- This test booklet has SEVEN QUESTIONS, totaling 150 points for the whole test. You have 120 minutes to do this test. **Plan your time well. Read the questions carefully.**
- This test is **closed book, closed note, with no electronic devices**.
- In order to get full credits, you need to give **correct** and **simplified** answers and explain in a **comprehensible way** how you arrive at them.
- **As a rule of thumb, you should give explicit and useful answers.** No points will be given for just writing down some generically true statements. In other words, your answers should try to make use of all the information given in the question.
- **As a rule of thumb, you should only use those methods that have been covered in class.** If you use some other methods "for the sake of convenience", at our discretion, we might not give you any credit. You have the right to contest. In that event, **you will be asked to explain your answer using only what has been covered in class up to the point of time of this exam.**

Name: Answer Key (Major:)

Question	Score
1.(20 pts)	
2.(20 pts)	
3.(20 pts)	
4.(20 pts)	
5.(20 pts)	
6.(20 pts)	
7.(30 pts)	
Total (150 pts)	

1. Consider the matrix $A = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 2 & 1 & 0 \\ -1 & 1 & 2 & -2 \\ 2 & 0 & -3 & 4 \end{pmatrix}$

(a) Find a basis for Col(A), Row(A), and Null(A).

(b) Is the linear transformation $X \longrightarrow AX$ onto?
If not, find a Y such that there is no X that satisfies $Y = AX$.

(c) Is the linear transformation $X \longrightarrow AX$ one-to-one?
If not, find X_1, X_2 such that $AX_1 = AX_2$ but $X_1 \neq X_2$.

(a)
$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ -1 & 1 & 2 & -2 & 0 \\ 2 & 0 & -3 & 4 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & -2 & -1 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & -\frac{3}{2} & 2 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\text{Col}(A) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$\text{Row}(A) = \text{Span}\left\{ \begin{pmatrix} 1 & 0 & -\frac{3}{2} & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & \frac{1}{2} & 0 \end{pmatrix} \right\}$$

$$\text{Null}(A) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{3}{2}\alpha - 2\beta \\ -\frac{\alpha}{2} \\ \alpha \\ \beta \end{pmatrix} \right\}$$

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$$= \alpha \begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\mathrm{Null}(A) = \mathrm{Span}\left\{ \begin{pmatrix} 3 \\ -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

(b) $X \longrightarrow AX$ is not onto as $\mathrm{Rank}(A) = 2 < 4$

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & a \\ 0 & 2 & 1 & 0 & b \\ -1 & 1 & 2 & -2 & c \\ 2 & 0 & -3 & 4 & d \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & a \\ 0 & 2 & 1 & 0 & b \\ 0 & 2 & 1 & 0 & a+c \\ 0 & -2 & -1 & 0 & d-2a \end{array}\right)$$

Y

In order for $AX = Y$ to be solvable, we need

$$\begin{cases} b = a+c \text{ and} \\ a+c = -(d-2a) \end{cases}$$

$$\Rightarrow \begin{cases} a+c-b = 0 \\ a-c-d = 0 \end{cases}$$

$$\boxed{\text{choose e.g. } Y = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}}$$

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(c) $X \longrightarrow AX$ is not one-to-one as there are free variables.

e.g. $\begin{pmatrix} 3 \\ -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in Null(A)$,

$$A\begin{pmatrix} 3 \\ -1 \\ 2 \\ 0 \end{pmatrix} = A\begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix} = A\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

X_1 X_2 X_3 are all different

but $AX_1 = AX_2 = AX_3 = 0$

2. Consider the subspace $S = \left\{ p(x) : \int_0^1 p(x)\,dx = 0 \right\}$ of $P_3 = \{p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3\}$, the space of polynomials of degree at most 3.

   (a) Find a basis for S.

   (b) Does the polynomial $p(x) = -1 + 4x - 6x^2 + 4x^3$ belong to S? If so, write $p(x)$ as a linear combination of the basis you have found for S.

(a) $$\int_0^1 (a_0 + a_1 x + a_2 x^2 + a_3 x^3)\,dx = 0$$

$$a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \frac{a_3}{4} = 0$$

$a_1 = \alpha$, $a_2 = \beta$, $a_3 = \gamma$, free vars.

$$a_0 = -\frac{\alpha}{2} - \frac{\beta}{3} - \frac{\gamma}{4}$$

$$p(x) = \left(-\frac{\alpha}{2} - \frac{\beta}{3} - \frac{\gamma}{4}\right) + \alpha x + \beta x^2 + \gamma x^3$$

$$= \alpha\left(-\frac{1}{2} + x\right) + \beta\left(-\frac{1}{3} + x^2\right) + \gamma\left(-\frac{1}{4} + x^3\right)$$

$$S = \text{Span}\left\{ \underbrace{-\frac{1}{2} + x}_{p_1(x)},\ \underbrace{-\frac{1}{3} + x^2}_{p_2(x)},\ \underbrace{-\frac{1}{4} + x^3}_{p_3(x)} \right\} \leftarrow \text{basis}$$

(b) $$-1 + \frac{4}{2} + \frac{(-6)}{3} + \frac{4}{4} = -1 + 2 - 2 + 1 = 0$$

Hence $p(x) \in S$

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Write $p(x) = c_1 p_1(x) + c_2 p_2(x) + c_3 p_3(x) + c_4 p_4(x)$

$$-1 + 4x - 6x^2 + 4x^3 = c_1\left(-\frac{1}{2} + x\right) + c_2\left(-\frac{1}{3} + x^2\right) + c_3\left(-\frac{1}{4} + x^3\right)$$

$c_3 = 4$

$c_2 = -6$

$c_1 = 4$

$$\boxed{\left(-1 + 4x - 6x^2 + 4x^3\right) = 4\left(-\frac{1}{2} + x\right) - 6\left(-\frac{1}{3} + x^2\right) + 4\left(-\frac{1}{4} + x^3\right)}$$

3. Consider the subspace of 2×2 matrices $S = \left\{ A : A \begin{pmatrix} 1 & 2 \\ 1 & -5 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & -5 \end{pmatrix} A \right\}$.

(a) Find a basis for S.

(b) Is $B = \begin{pmatrix} 5 & 4 \\ 2 & -7 \end{pmatrix}$ from S? If so, write B as a linear combination of the basis you have found for S.

(a) $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & -5 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & -5 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{pmatrix} a+b & 2a-5b \\ c+d & 2c-5d \end{pmatrix} = \begin{pmatrix} a+2c & b+2d \\ a-5c & b-5d \end{pmatrix}$$

$$\left.\begin{aligned} a+b = a+2c &\Rightarrow \underline{b = 2c} \\ 2a-5b = b+2d &\Rightarrow 2a-6b-2d=0 \\ & \underline{a-3b-d=0} \\ c+d = a-5c &\Rightarrow \underline{a-6c-d=0} \\ 2c-5d = b-5d &\Rightarrow \underline{b=2c} \end{aligned}\right\} \Leftrightarrow \begin{aligned} &b = 2c \\ &a = 6c+d \\ &c, d \text{- free} \end{aligned}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 6c+d & 2c \\ c & d \end{pmatrix} = c \begin{pmatrix} 6 & 2 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\boxed{S = \operatorname{Span}\left\{ \begin{pmatrix} 6 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}}$$

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(b) Try to write:

$$\begin{pmatrix} 5 & 4 \\ 2 & -7 \end{pmatrix} = c_1 \begin{pmatrix} 6 & 2 \\ 1 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$c_1 = 2$ $\qquad$ $c_2 = -7$

$$\boxed{\begin{pmatrix} 5 & 4 \\ 2 & -7 \end{pmatrix} = 2 \begin{pmatrix} 6 & 2 \\ 1 & 0 \end{pmatrix} - 7 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

Hence $\begin{pmatrix} 5 & 4 \\ 2 & -7 \end{pmatrix} \in S$

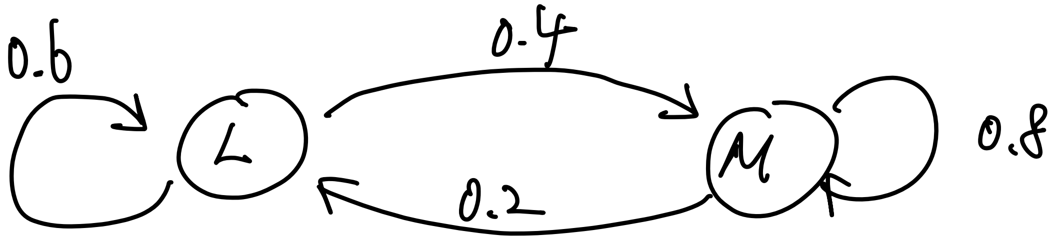
 

GO PDCBRS!

4. The Purdue Daily Campus Bike Rental Service (PDCBRS) has two pick up and drop off sites: L (Lily Hall) and M (MSEE), i.e. you can only pick up and drop off at L or M (so that there will not be bikes scattered all over the campus). A bike picked up from L has a 60% chance of being returned to L (and hence 40% chance being dropped off at M). A bike picked up from M has a 80% chance of being returned to M (and hence 20% chance being dropped off at L).

   (a) Find the equilibrium distribution of bikes at L and M.

   (b) Of course the actual daily number of bikes at L and M can fluctuate from the equilibrium distribution. Suppose on Oct. 1st, 2024, 90% of the bikes were at M (and hence 10% of the bikes were at L). Find the distributions of bikes at L and M one day, two day and n-day afterward.



(a) Transition matrix $= \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix} = A$

Find X s.t. $AX = X \iff (A - I)X = 0$

$$\left(\begin{array}{cc|c} -0.4 & 0.2 & 0 \\ +0.4 & -0.2 & 0 \end{array}\right) = \left(\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow X = \begin{pmatrix} \frac{\alpha}{2} \\ \alpha \end{pmatrix}$$

Choose α s.t. $\frac{\alpha}{2} + \alpha = 1 \Rightarrow \alpha = \frac{2}{3} \Rightarrow X = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} \approx \begin{pmatrix} 33.3\% \\ 66.7\% \end{pmatrix}$

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(b) $\begin{pmatrix} a_0 = 0.1 \\ b_0 = 0.9 \end{pmatrix}$, $\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{pmatrix}\begin{pmatrix} 0.1 \\ 0.9 \end{pmatrix} = \begin{pmatrix} 0.24 \\ 0.76 \end{pmatrix}$

$$\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{pmatrix}\begin{pmatrix} 0.24 \\ 0.76 \end{pmatrix} = \begin{pmatrix} 0.296 \\ 0.704 \end{pmatrix}$$

For n-day distribution, diagonalize A:

$$\det(A-\lambda I) = \det\begin{pmatrix} 0.6-\lambda & 0.2 \\ 0.4 & 0.8-\lambda \end{pmatrix} = \lambda^2 - 0.14\lambda + 0.48 - 0.08$$
$$= \lambda^2 - 0.14\lambda + 0.4$$
$$= (\lambda-1)(\lambda-0.4)$$

$\lambda_1 = 1 \Rightarrow$ Steady state, $X_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $(a=2)$

$\lambda_1 = 0.4 \Rightarrow (A-0.4I\,|\,0) \Rightarrow \left(\begin{array}{cc|c} 0.2 & 0.2 & 0 \\ 0.4 & 0.4 & 0 \end{array}\right) \Rightarrow X_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$A^n = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & (0.4)^n \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & (0.4)^n \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}\frac{1}{3}$$

$$= \begin{pmatrix} 1 & -(0.4)^2 \\ 2 & (0.4)^2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}\frac{1}{3} = \frac{1}{3}\begin{pmatrix} 1+2(0.4)^n & 1-(0.4)^n \\ 2-2(0.4)^n & 2+(0.4)^n \end{pmatrix}$$

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$$\begin{pmatrix} a_n \\ b_n \end{pmatrix} = A^n \begin{pmatrix} 0.1 \\ 0.9 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1+2(0.4)^n & 1-(0.4)^n \\ 2-2(0.4)^n & 2+(0.4)^n \end{pmatrix}\begin{pmatrix} 0.1 \\ 0.9 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 0.1+0.2(0.4)^n+0.9-0.9(0.4)^n \\ 0.2-0.2(0.4)^n+1.8+0.9(0.4)^n \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 1-0.7(0.4)^n \\ 2+0.7(0.4)^n \end{pmatrix}$$

$$\left.\begin{aligned} a_n &= \frac{1}{3}(1-0.7(0.4)^n) \\ b_n &= \frac{1}{3}(2+0.7(0.4)^n) \end{aligned}\right\} \quad (\text{Note: } a_n+b_n=1.)$$

5. In class, we often consider problems such as given a matrix A, find its eigenvalues and eigenvectors. This question tackles the "reverse problem".

   (a) Consider a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Find or describe all those numbers a, b, c, d such that the matrix A has eigenvalue 2 and 3. Give a couple of explicit but *different* examples of a, b, c, d. Effectively, how many free variables does this problem have?

   (Hint: how does the characteristic polynomial of A, $\det(A - \lambda I)$ look like?)

   (b) Write down a 3×3 matrix A such that A has eigenvalues $1, 1, 3$ and the corresponding eigenvectors are $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$.

   How many such matrix can you find?

(a) $\det(A - \lambda I) = \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = (a-\lambda)(d-\lambda) - bc$

$= \lambda^2 - (a+d)\lambda + ad - bc$

$(\lambda_1 = 2, \lambda_2 = 3 \Rightarrow) = (\lambda-2)(\lambda-3)$

$= \lambda^2 - 5\lambda + 6$

Hence $a + d = 5$

$ad - bc = 6$

(i) $d = 0,\ a = 5,\ bc = -6 \Rightarrow b = 2,\ c = -3$

$$\begin{pmatrix} 5 & 2 \\ -3 & 0 \end{pmatrix}$$

(ii) $a = 0,\ d = 5,\ bc = -6 \Rightarrow b = 2,\ c = -3$

$$\begin{pmatrix} 0 & 2 \\ -3 & 5 \end{pmatrix}$$

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$a+d=5$
$ad-bc=6$

2 eqns in 4 unknowns $\Rightarrow$ 2 free variables.

(b) $A=QDQ^{-1}$

M1

$$= \begin{pmatrix} 1 & 0 & 3 \\ 1 & 0 & 0 \\ 2 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & & 3 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 & 3 \\ 1 & 0 & 0 \\ 2 & 1 & 4 \end{pmatrix}^{-1}}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 2 & 1 & 4 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & -3 & -1 & 1 & 0 \\ 0 & 1 & -2 & -2 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 0 & 1 \\ 0 & 0 & 1 & 1/3 & -1/3 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -4/3 & -2/3 & 1 \\ 0 & 0 & 1 & 1/3 & -1/3 & 0 \end{array}\right)$$

$$= \begin{pmatrix} 1 & 0 & 9 \\ 1 & 0 & 0 \\ 2 & 1 & 12 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -4/3 & -2/3 & 1 \\ 1/3 & -1/3 & 0 \end{pmatrix} = \begin{pmatrix} 3 & -2 & 0 \\ 0 & 1 & 0 \\ 8/3 & -\frac{8}{3} & 1 \end{pmatrix}$$

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M2 Can also solve for $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \quad \ldots$$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \ldots$$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} = 3 \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 9 \\ 0 \\ 12 \end{pmatrix} \quad \ldots$$

6. Consider the matrix $P = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$. (The origin of this matrix comes from projecting vectors in $\mathbb{R}^3$ onto the plane $x + y + z = 0$.)

(a) Show that $P^2 = P$ and hence show that the eigenvalues of P *must be* 0 or 1.

(b) Find the eigevectors for P.

(c) What are the geometric and algebraic multiplicities of the eigenvalues of P?

(a)

$$P^2 = \frac{1}{9}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 6 & -3 & -3 \\ -3 & 6 & -3 \\ -3 & -3 & 6 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} = P$$

$$PX = \lambda X \Rightarrow P^2 X = \lambda P X = \underline{\lambda^2} X$$

P

$$\overset{\parallel}{PX} = \underline{\lambda} X$$

$\lambda^2 = \lambda$

$\Rightarrow \lambda = 0, 1$

(b) $\lambda = 0 \Rightarrow PX = 0$

$$\left(\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 2 & -1 & -1 & 0 \\ -1 & -1 & 2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_3 = \alpha$

$x_2 = \alpha$

$x_1 = \alpha$

$\alpha = 1$

$$X_1 = \begin{pmatrix} \alpha \\ \alpha \\ \alpha \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{X_1}$$

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$\lambda = 1 \Rightarrow (P - I)X = 0$

$$\left(\begin{array}{ccc|c} -1/3 & -1/3 & -1/3 & 0 \\ -1/3 & -1/3 & -1/3 & 0 \\ -1/3 & -1/3 & -1/3 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \begin{array}{l} x_2 = \alpha \\ x_3 = \beta \\ x_1 = -\alpha - \beta \end{array}$$

$$X = \begin{pmatrix} -\alpha - \beta \\ \alpha \\ \beta \end{pmatrix} = \alpha \underbrace{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}_{X_2} + \beta \underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}_{X_3}$$

(c) $\lambda_1 = 0 \quad 1 \leq \underline{g_1 = 1} \leq m_1 \quad \Rightarrow$

$\lambda_2 = 1 \quad 1 \leq \underline{g_2 = 2} \leq m_2$

$1 \leq m_1, \quad 2 \leq m_2$

But $\underline{m_1 + m_2 = 3} \Rightarrow \underline{m_1 = 1, \ m_2 = 2}$

7. For the following, let $A^{(n\times n)}$ be a Markov transition matrix, i.e. all its entries are non-negative and the sum of each column is one. (Note: the non-negativity of the entries is also important for the general theory of Markov Chain but is not relevant for this problem here.)

   (a) Let X be a (column) vector such that the sum of all its entries equals one. Show that $Y = AX$ is a also vector such that sum of all its entries equals one.

   (b) Let B be another transition matrix. Show that AB is also a transition matrix.

   (c) Consider the transpose A^T of A and the vector $X = (1, 1, \ldots, 1)^T$, i.e. X is a (column) vector with all entries being one. Show that $A^T X = X$, i.e. X is an eigenvector with eigenvalue one for A^T.

   (d) (Hence) show that $\lambda = 1$ is also an eigenvalue of A.

   (Hint: how do the characteristic polynomials $\det(A - \lambda I)$ and $\det(A^T - \lambda I)$ relate to each other?)

(a)
$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{pmatrix}$$

$+$

$$\begin{aligned} &= \left(\cancelto{1}{a_{11} + a_{21} + \cdots + a_{n1}}\right)x_1 \\ &\quad + \left(\cancelto{1}{a_{12} + a_{22} + \cdots + a_{n2}}\right)x_2 \\ &\quad\ \vdots \\ &\quad + \left(\cancelto{1}{a_{1n} + a_{2n} + \cdots + a_{nn}}\right)x_n \\ &= x_1 + x_2 + \cdots + x_n = 1 \end{aligned}$$

(b) $AB = A\,[v_1\ v_2 \cdots v_n]$ ← each of v_i has sum $= 1$

$= [Av_1\ Av_2 \cdots Av_n]$ ← by (a), each of Av_i has sum $= 1$

Hence AB is a transition matrix

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cols of A

(c) $$A^T X = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

sum of each col. of A.

$$= \begin{pmatrix} a_{11}+a_{21}+\cdots+a_{n1} \\ a_{12}+a_{22}+\cdots+a_{n2} \\ \vdots \\ a_{1n}+a_{2n}+\cdots+a_{nn} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = X$$

i.e. $A^T X = 1 X$

$\lambda = 1$ is an eigenvalue of A^T

$X = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ is an eigenvector of A^T

(d) $$\det(A^T - \lambda I) = \det((A - \lambda I)^T) = \det(A - \lambda I)$$

$(\det M^T = \det M)$

i.e. A^T and A have the same char. poly.

So A^T and A have the same set of eigenvalues.

$\lambda = 1 \Rightarrow \lambda = 1$ is also an eigenvalue of A.