MA 351: Introduction to Linear Algebra and Its Applications
Fall 2024, Midterm Two
Instructor: Yip

- This test booklet has FIVE QUESTIONS, totaling 120 points for the whole test. You have 75 minutes to do this test. **Plan your time well. Read the questions carefully.**
- This test is **closed book, closed note, with no electronic devices**.
- In order to get full credits, you need to give **correct** and **simplified** answers and explain in a **comprehensible way** how you arrive at them.
- **As a rule of thumb, you should give explicit and useful answers.** No points will be given for just writing down some generically true statements. In other words, your answers should try to make use of all the information given in the question.
- **As a rule of thumb, you should only use those methods that have been covered in class.** If you use some other methods "for the sake of convenience", at our discretion, we might not give you any credit. You have the right to contest. In that event, **you will be asked to explain your answer using only what has been covered in class up to the point of time of this exam.**

Name: Answer Key (Major:)

Question	Score
1.(20 pts)	
2.(20 pts)	
3.(20 pts)	
4.(20 pts)	
5.(40 pts)	
Total (120 pts)	

both square, 3x3

$$AB = I \iff BA = I$$

1. You are given the following information:

$$\underbrace{A\begin{pmatrix}1&0&1\\2&1&1\\-1&0&0\end{pmatrix}}_{B} = \underbrace{\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}}_{I}.$$

Fill in blanks:

$$A\boxed{?} = \begin{pmatrix}1&0\\2&2\\3&1\end{pmatrix}, \quad \text{and} \quad \boxed{?}\,A = \begin{pmatrix}1&0&2\\1&1&1\end{pmatrix}.$$

B

B

$$I\ \cancel{BA}\,\boxed{?} = B\begin{pmatrix}1&0\\2&2\\3&1\end{pmatrix}$$

$$\boxed{?} = \begin{pmatrix}1&0&1\\2&1&1\\-1&0&0\end{pmatrix}\begin{pmatrix}1&0\\2&2\\3&1\end{pmatrix} = \boxed{\begin{pmatrix}4&1\\7&3\\-1&0\end{pmatrix}}$$

$$\boxed{?}\,\overset{I}{\cancel{AB}} = \begin{pmatrix}1&0&2\\1&1&1\end{pmatrix}B$$

$$\boxed{?} = \begin{pmatrix}1&0&2\\1&1&1\end{pmatrix}\begin{pmatrix}1&0&1\\2&1&1\\-1&0&0\end{pmatrix} = \boxed{\begin{pmatrix}-1&0&1\\2&1&2\end{pmatrix}}$$

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2. Find the inverse of the following two matrices:

$$A = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix}.$$

$$\left[\begin{array}{ccc|ccc} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & a & 0 & 1 & 0 & -b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -a & ac-b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\underbrace{\qquad\qquad}_{A^{-1}}$$

$$\boxed{A^{-1} = \begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}}$$

Note : $B = A^T$

Hence $B^{-1} = (A^T)^{-1} = (A^{-1})^T = \boxed{\begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ ac-b & -c & 1 \end{bmatrix}}$

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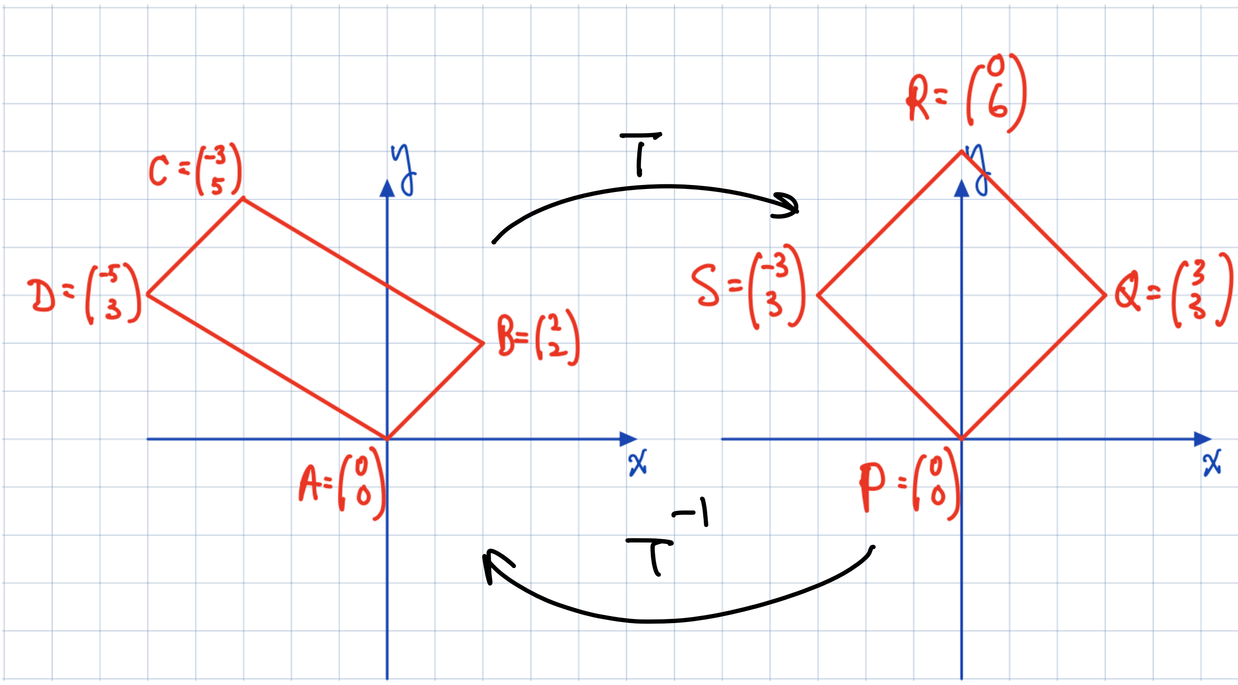
3. Consider the following figure:



(a) Find the linear transformation that maps the polygon $ABCD$ to $PQRS$.

(b) Find also the linear transformation that maps the polygon $PQRS$ to $ABCD$.

Note: $ABCD$ and $PQRS$ are parallelograms.

$C = B + D$ and $R = Q + S$

(a) Find T, linear, such that

$T(A) = P$, $\quad T(B) = Q$, $\quad T(D) = S$

automatic : $T(0) = 0$

(automatically $T(C) = T(B+D) = T(B) + T(D)$
$= Q + S = R$

$$T \begin{bmatrix} 2 & -5 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 3 & 3 \end{bmatrix}$$

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$$T = \begin{bmatrix} 3 & -3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 2 & -5 \\ 2 & 3 \end{bmatrix}^{-1} = \frac{\begin{bmatrix} 3 & -3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ -2 & 2 \end{bmatrix}}{2(3)-(2)(-5)}$$

$$= \boxed{\frac{\begin{bmatrix} 15 & 9 \\ 3 & 21 \end{bmatrix}}{16}}$$

(b) $$T^{-1} = \left(\frac{1}{16} \begin{bmatrix} 15 & 9 \\ 3 & 21 \end{bmatrix} \right)^{-1}$$

$$\begin{array}{r} 15 \\ 21 \\ \hline 30 \\ 15 \end{array}$$

$$= \frac{16 \begin{bmatrix} 21 & -9 \\ -3 & 15 \end{bmatrix}}{15(21) - 27}$$

$$\begin{array}{r} 315 \\ 27 \\ \hline 288 \end{array}$$

$$= \frac{16}{288} \begin{bmatrix} 21 & -9 \\ -3 & 15 \end{bmatrix}$$

$$= \frac{16}{96} \begin{bmatrix} 7 & -3 \\ -1 & 5 \end{bmatrix} = \boxed{\frac{1}{6} \begin{bmatrix} 7 & -3 \\ -1 & 5 \end{bmatrix}}$$

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4. Let X_1, X_2 and X_3 be three linearly independent vectors in some vector space. Consider the following three vectors:

$$\begin{aligned} Y_1 &= X_1 + X_3; \\ Y_2 &= 2X_1 - X_2; \\ Y_3 &= X_2 + 3X_3. \end{aligned}$$

Is $\{Y_1, Y_2, Y_3\}$ a basis for $\mathrm{Span}\{X_1, X_2, X_3\}$? If so, write a general vector from $\mathrm{Span}\{X_1, X_2, X_3\}$ as a linear combination of Y_1, Y_2, Y_3.

$\dim(\mathrm{Span}\{X_1, X_2, X_3\}) = 3$

Check for span of $\{Y_1, Y_2, Y_3\}$

For any $aX_1 + bX_2 + cX_3$, try to write it as $c_1Y_1 + c_2Y_2 + c_3Y_3$

i.e. $c_1(X_1 + X_3) + c_2(2X_1 - X_2) + c_3(X_2 + 3X_3) = aX_1 + bX_2 + cX_3$

Coeff. of X_1 : $c_1 + 2c_2 = a$

Coeff. of X_2 : $-c_2 + c_3 = b$

Coeff. of X_3 : $c_1 + 3c_3 = c$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & a \\ 0 & -1 & 1 & b \\ 1 & 0 & 3 & c \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & a \\ 0 & 1 & -1 & -b \\ 0 & -2 & 3 & c-a \end{array}\right]$$

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$$\to \left[\begin{array}{ccc|c} 1 & 2 & 0 & a \\ 0 & 1 & -1 & -b \\ 0 & 0 & 1 & -2b+c-a \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 2 & 0 & a \\ 0 & 1 & 0 & -a-3b+c \\ 0 & 0 & 1 & -a-2b+c \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3a+6b-2c \\ 0 & 1 & 0 & -a-3b+c \\ 0 & 0 & 1 & -a-2b+c \end{array}\right]$$

Hence

$$(3a+6b-2c)Y_1 + (-a-3b+c)Y_2 + (-a-2b+c)Y_3$$
$$= aX_1 + bX_2 + cX_3$$

(Just to check:

$$(3a+6b-2c)(X_1+X_3) + (-a-3b+c)(2X_1 - X_2)$$
$$+ (-a-2b+c)(X_2 + 3X_3) = aX_1 + bX_2 + cX_3$$
)

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Hence $\underbrace{\{Y_1, Y_2, Y_3\}}_{3 \text{ vectors}}$ spans the whole of $\underbrace{\text{Span}\{X_1, X_2, X_3\}}_{\dim 3}$

By Thm 2.8, $\{Y_1, Y_2, Y_3\}$ is automatically lin ind.
Hence $\{Y_1, Y_2, Y_3\}$ is a basis for $\text{Span}\{X_1, X_2, X_3\}$

5. Consider the following matrix:

$$A = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ 1 & -2 & -1 & 0 \end{pmatrix}$$

(a) Find a basis for Col(A), Row(A) and Null(A).

(b) Write each column of A as a linear combination of the basis vectors you have found for Col(A).

(c) Write each row of A as a linear combination of the basis vectors you have found for Row(A).

(d) Consider the combined collection of the basis vectors from Null(A) and Row(A). (Just for consistency, write all the vectors as column vectors.)

Does it form a basis for R^4?

(a) $A = \left(\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 2 & 0 \\ 1 & -2 & -1 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 \end{array}\right)$

$\to \left(\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

Basis for Col(A) $= \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix} \right\}$

Basis for Row(A) $= \left\{ \begin{pmatrix} 1 & 0 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 1 \end{pmatrix} \right\}$

Solve for $AX = 0$: $\quad x_3 = \alpha, \; x_4 = \beta$, free

$x_1 = -\alpha - 2\beta, \; x_2 = -\alpha - \beta$

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$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -\alpha-2\beta \\ -\alpha-\beta \\ \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

Basis for $Null(A) = \left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

(b) $AX=0$ gives

$$x_1 \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{u_1} + x_2 \underbrace{\begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix}}_{u_2} + x_3 \underbrace{\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}}_{u_3} + x_4 \underbrace{\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}}_{u_4} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$(-\alpha-2\beta)u_1 + (-\alpha-\beta)u_2 + \alpha u_3 + \beta u_4 = 0$$

$\alpha=1, \beta=0 \implies$

$\alpha=0, \beta=1 \implies$

$$\boxed{\begin{aligned} u_3 &= u_1 + u_2 \\ u_4 &= 2u_1 + u_2 \\ u_1 &= 1u_1 + 0u_2 \\ u_2 &= 0u_1 + 1u_2 \end{aligned}}$$

Basis vector for Row(A)

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(c)

$$(1 \quad -1 \quad 0 \quad 1) = 1\,(1 \quad 0 \quad 1 \quad 2) + (-1)\,(0 \quad 1 \quad 1 \quad 1)$$

$$(1 \quad 0 \quad 1 \quad 2) = 1\,(1 \quad 0 \quad 1 \quad 2) + 0\,(0 \quad 1 \quad 1 \quad 1)$$

$$(1 \quad -2 \quad -1 \quad 0) = 1\,(1 \quad 0 \quad 1 \quad 2) + (-2)\,(0 \quad 1 \quad 1 \quad 1)$$

(d)

$$\begin{array}{c} \left[\begin{array}{cccc|c} -1 & -2 & 1 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 \end{array}\right] \\ \text{Null} \quad \underbrace{V_1 \quad V_2} \quad \underbrace{V_3 \quad V_4} \quad \text{Row} \end{array}$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ -1 & -2 & 1 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & -2 & 2 & 1 & 0 \\ 0 & -1 & 1 & 2 & 0 \\ 0 & 1 & 2 & 1 & 0 \end{array}\right]$$

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$$\to \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & -1 & 1 & 2 & 0 \\ 0 & -2 & 2 & 1 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 3 & 3 & 0 \\ 0 & 0 & 6 & 3 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 \end{array}\right]$$

no free var. $\Rightarrow v_1, v_2, v_3, v_4$ lin ind.

As $\mathbb{R}^4$ has dim $= 4$.

$\Rightarrow \{v_1, v_2, v_3, v_4\}$ is a basis for $\mathbb{R}^4$.

(Thm 2.8)

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