MA 351, Fall 2024, Quiz 1–5 Solution

Quiz #1 Linearity Property of Linear Transformation

$$\text{input}\ \begin{pmatrix}x\\y\end{pmatrix} \longrightarrow \boxed{\;T\;} \longrightarrow \begin{pmatrix}p\\q\end{pmatrix} = \begin{pmatrix}2x-y\\x+3y\end{pmatrix}\ \text{output}$$

$$\begin{pmatrix}1\\2\end{pmatrix} \longrightarrow \boxed{\;T\;} \longrightarrow \begin{pmatrix}p\\q\end{pmatrix} = \begin{pmatrix}2(1)-2\\1+3(2)\end{pmatrix} = \begin{pmatrix}0\\7\end{pmatrix}$$

$+$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $+$

$$\begin{pmatrix}-1\\7\end{pmatrix} \longrightarrow \boxed{\;T\;} \longrightarrow \begin{pmatrix}p\\q\end{pmatrix} = \begin{pmatrix}2(-1)-7\\-1+3(7)\end{pmatrix} = \begin{pmatrix}-9\\20\end{pmatrix}$$

$$\begin{pmatrix}0\\9\end{pmatrix} \longrightarrow \boxed{\;T\;} \longrightarrow \begin{pmatrix}p\\q\end{pmatrix} = \begin{pmatrix}2(0)-9\\0+3(9)\end{pmatrix} = \begin{pmatrix}-9\\27\end{pmatrix}$$

input $\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \boxed{T} \longrightarrow \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 2x-y \\ x+3y \end{pmatrix}$ output

Prove: If $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \longrightarrow \boxed{T} \longrightarrow \begin{pmatrix} p_1 \\ q_1 \end{pmatrix}$,

$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \longrightarrow \boxed{T} \longrightarrow \begin{pmatrix} p_2 \\ q_2 \end{pmatrix}$,

then $\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \end{pmatrix} \longrightarrow \boxed{T} \longrightarrow \begin{pmatrix} p_1+p_2 \\ q_1+q_2 \end{pmatrix}$

(addition of inputs leads to addition of outputs)

More generally: (Hw 1. Add. Problem)

$$\boxed{T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}}$$

(a) $$T\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) = T\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \end{pmatrix}$$

$$= \begin{pmatrix} a(x_1+x_2) + b(y_1+y_2) \\ c(x_1+x_2) + d(y_1+y_2) \end{pmatrix} = \begin{pmatrix} ax_1 + ax_2 + by_1 + by_2 \\ cx_1 + cx_2 + dy_1 + dy_2 \end{pmatrix}$$

$$= \begin{pmatrix} ax_1+by_1 \\ cx_1+dy_1 \end{pmatrix} + \begin{pmatrix} ax_2+by_2 \\ cx_2+dy_2 \end{pmatrix} = T\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + T\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

(b) $$T\left(\lambda\begin{pmatrix}x\\y\end{pmatrix}\right)=T\begin{pmatrix}\lambda x\\\lambda y\end{pmatrix}=\begin{pmatrix}a(\lambda x)+b(\lambda y)\\c(\lambda x)+d(\lambda y)\end{pmatrix}$$

$$=\lambda\begin{pmatrix}ax+by\\cx+dy\end{pmatrix}=\lambda T\begin{pmatrix}x\\y\end{pmatrix}$$

(c) $$T\left(\lambda\begin{pmatrix}x_1\\y_1\end{pmatrix}+\mu\begin{pmatrix}x_2\\y_2\end{pmatrix}\right)=T\begin{pmatrix}\lambda x_1+\mu x_2\\\lambda y_1+\mu y_2\end{pmatrix}$$

M1

$$=\begin{pmatrix}a(\lambda x_1+\mu x_2)+b(\lambda y_1+\mu y_2)\\c(\lambda x_1+\mu x_2)+d(\lambda y_1+\mu y_2)\end{pmatrix}$$

$$=\begin{pmatrix}a\lambda x_1+b\lambda y_1\\c\lambda x_1+d\lambda y_1\end{pmatrix}+\begin{pmatrix}a\mu x_2+b\mu y_2\\c\mu x_2+d\mu y_2\end{pmatrix}$$

$$= \lambda \begin{pmatrix} ax_1 + by_1 \\ cx_1 + dy_1 \end{pmatrix} + \mu \begin{pmatrix} ax_2 + by_2 \\ cx_2 + dy_2 \end{pmatrix}$$

$$= \lambda T\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \mu T\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

M2 (Make use of a & b)

$$T\left(\lambda\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \mu\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) \overset{a}{=} T\left(\lambda\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right) + T\left(\mu\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right)$$

$$\overset{b}{=} \lambda T\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \mu T\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

$$\boxed{T(\alpha \vec{u} + \beta \vec{v}) = \alpha T(\vec{u}) + \beta T(\vec{v})}$$

(Quiz #2) (Penney, p.57, Ex 1.11)

(a) Find conditions on a, b, c, d s.t.

$$\begin{cases} x+y+2z+w=a \\ 3x-4y+z+w=b \\ 4x-3y+3z+2w=c \\ 5x-2y+5z+3w=d \end{cases}$$

is solvable.

(b) If $b=2$, $c=-1$, find all a and d such that the above system is solvable.

(a)

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & a \\ 3 & -4 & 1 & 1 & b \\ 4 & -3 & 3 & 2 & c \\ 5 & -2 & 5 & 3 & d \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & a \\ 0 & -7 & -5 & -2 & b-3a \\ 0 & -7 & -5 & -2 & c-4a \\ 0 & -7 & -5 & -2 & d-5a \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & a \\ 0 & -7 & -5 & -2 & b-3a \\ 0 & 0 & 0 & 0 & c-a-b \\ 0 & 0 & 0 & 0 & d-2a-b \end{array}\right] \begin{array}{l} \\ \\ \leftarrow \quad c = a+b \\ \leftarrow \quad d = 2a+b \end{array}$$

(b) $b = 2, \quad c = -1 \Rightarrow \quad -1 = a + 2 \Rightarrow a = -3$

$d = 2(-3) + 2 \Rightarrow d = -4$

Quiz #3

Consider $\begin{cases} 7x - y = \lambda x \\ -6x + 8y = \lambda y \end{cases}$

Solve the system for $\lambda = 5,\ 10,\ 15$

$\lambda = 5$: $\begin{cases} 2x - y = 0 \\ -6x + 3y = 0 \end{cases} \Longrightarrow \left(\begin{array}{cc|c} 2 & -1 & 0 \\ -6 & 3 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$y = s,\ x = \frac{s}{2}$

Solution: $\begin{pmatrix} x \\ y \end{pmatrix} = s \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$ (infinitely many solution)

$\underline{\lambda = 10}$

$$\begin{cases} -3x - y = 0 \\ -6x - 2y = 0 \end{cases}$$

$$\left(\begin{array}{cc|c} -3 & -1 & 0 \\ -6 & -2 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} -3 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$y = s, \quad x = -\frac{1}{3}s$$

Solution : $\begin{pmatrix} x \\ y \end{pmatrix} = s\begin{pmatrix} -1/3 \\ 1 \end{pmatrix}$ (inf. many solutions)

$\underline{\lambda = 15}$

$$\begin{cases} -8x - y = 0 \\ -6x - 7y = 0 \end{cases}$$

$$\left(\begin{array}{cc|c} -8 & -1 & 0 \\ -6 & -7 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 8 & 1 & 0 \\ 6 & 7 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cc|c} 1 & 1/8 & 0 \\ 6 & 7 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & \frac{1}{8} & 0 \\ 0 & \frac{50}{8} & 0 \end{array}\right)$$

Solution: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$x = 0, \; y = 0$

(only the trivial soln)

Quiz #4 Fill in blanks

(a) $$\begin{pmatrix}1\\0\\0\end{pmatrix} + s\begin{pmatrix}-3\\1\\1\end{pmatrix} + t\begin{pmatrix}-1\\0\\1\end{pmatrix} = \begin{pmatrix}-3\\1\\2\end{pmatrix} + \underbrace{\boxed{?}}_{p}\begin{pmatrix}-2\\1\\0\end{pmatrix} + \underbrace{\boxed{?}}_{q}\begin{pmatrix}5\\-2\\-1\end{pmatrix}$$

(b) $$\begin{pmatrix}1\\0\\0\end{pmatrix} + \overbrace{\boxed{?}}^{p}\begin{pmatrix}-3\\1\\1\end{pmatrix} + \overbrace{\boxed{?}}^{q}\begin{pmatrix}-1\\0\\1\end{pmatrix} = \begin{pmatrix}-3\\1\\2\end{pmatrix} + s\begin{pmatrix}-2\\1\\0\end{pmatrix} + t\begin{pmatrix}5\\-2\\-1\end{pmatrix}$$

a $$p\begin{pmatrix}-2\\1\\0\end{pmatrix} + q\begin{pmatrix}5\\-2\\-1\end{pmatrix} = \begin{pmatrix}4\\-1\\-2\end{pmatrix} + s\begin{pmatrix}-3\\1\\1\end{pmatrix} + t\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

$$\begin{pmatrix}-2 & 5\\1 & -2\\0 & -1\end{pmatrix}\begin{pmatrix}p\\q\end{pmatrix} = \begin{pmatrix}4-3s-t\\-1+s\\-2+s+t\end{pmatrix}$$

$$\left[\begin{array}{cc|c} -2 & 5 & 4-3s-t \\ 1 & -2 & -1+s \\ 0 & -1 & -2+s+t \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} 1 & -2 & -1+s \\ -2 & 5 & 4-3s-t \\ 0 & +1 & 2-s-t \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} 1 & -2 & -1+s \\ 0 & 1 & 2-s-t \\ 0 & 1 & 2-s-t \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} 1 & 0 & 3-s-2t \\ 0 & 1 & 2-s-t \\ 0 & 0 & 0\ \ 0\ \ 0 \end{array}\right] \nearrow \begin{array}{l} p = 3-s-2t \\ q = 2-s-t \end{array}$$

$$\begin{pmatrix}1\\0\\0\end{pmatrix} + s\begin{pmatrix}-3\\1\\1\end{pmatrix} + t\begin{pmatrix}-1\\0\\1\end{pmatrix} = \begin{pmatrix}-3\\1\\2\end{pmatrix} + \underbrace{\boxed{?}}_{3-s-2t}\begin{pmatrix}-2\\1\\0\end{pmatrix} + \underbrace{\boxed{?}}_{2-s-t}\begin{pmatrix}5\\-2\\-1\end{pmatrix}$$

Given $s, t \Longrightarrow p = 3-s-2t, \; q = 2-s-t$

(b) M1 You can in fact solve for s, t in terms of p, q

$$\left.\begin{aligned} p &= 3-s-2t \\ q &= 2-s-t \end{aligned}\right) \Longrightarrow \left\{\begin{aligned} s &= 1+p-2q \\ t &= 1-p+q \end{aligned}\right.$$

Hence

$$\begin{pmatrix}1\\0\\0\end{pmatrix} + \boxed{?}\begin{pmatrix}-3\\1\\1\end{pmatrix} + \boxed{?}\begin{pmatrix}-1\\0\\1\end{pmatrix} = \begin{pmatrix}-3\\1\\2\end{pmatrix} + p\begin{pmatrix}-2\\1\\0\end{pmatrix} + q\begin{pmatrix}5\\-2\\-1\end{pmatrix}$$

$s = 1+p-2q$ $\quad$ $t = 1-p+q$

or equivalently

$$\begin{pmatrix}1\\0\\0\end{pmatrix} + \boxed{?}\begin{pmatrix}-3\\1\\1\end{pmatrix} + \boxed{?}\begin{pmatrix}-1\\0\\1\end{pmatrix} = \begin{pmatrix}-3\\1\\2\end{pmatrix} + s\begin{pmatrix}-2\\1\\0\end{pmatrix} + t\begin{pmatrix}5\\-2\\-1\end{pmatrix}$$

$p = 1+s-2t,$ $\quad$ $q = 1-s+t$

Alternatively M2

$$\begin{pmatrix}1\\0\\0\end{pmatrix} + p\begin{pmatrix}-3\\1\\1\end{pmatrix} + q\begin{pmatrix}-1\\0\\1\end{pmatrix} = \begin{pmatrix}-3\\1\\2\end{pmatrix} + s\begin{pmatrix}-2\\1\\0\end{pmatrix} + t\begin{pmatrix}5\\-2\\-1\end{pmatrix}$$

$$p\begin{pmatrix}-3\\1\\1\end{pmatrix} + q\begin{pmatrix}-1\\0\\1\end{pmatrix} = \begin{pmatrix}-4-2s+5t\\1+s-2t\\2-t\end{pmatrix}$$

$$\left(\begin{array}{cc|c}-3 & -1 & -4-2s+5t\\1 & 0 & 1+s-2t\\1 & 1 & 2-t\end{array}\right)$$

$$\to \left(\begin{array}{cc|c} 1 & 0 & 1+s-2t \\ -3 & -1 & -4-2s+5t \\ 1 & 1 & 2-t \end{array}\right)$$

$$\to \left(\begin{array}{cc|c} 1 & 0 & 1+s-2t \\ 0 & -1 & -1+s-t \\ 0 & 1 & 1-s+t \end{array}\right) \quad \text{the same}$$

$$\left(\begin{array}{cc|c} 1 & 0 & 1+s-2t \\ 0 & 1 & 1-s+t \\ 0 & 0 & 0 \end{array}\right)$$

$$p = 1+s-2t, \quad q = 1-s+t$$

Quiz #5 (Penney p. 101 Ex 2.3)

Let $V = \mathrm{Span}\left\{ \overset{A}{\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}}, \overset{B}{\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}}, \overset{C}{\begin{pmatrix} -1 & -2 \\ -3 & -5 \end{pmatrix}}, \overset{D}{\begin{pmatrix} -1 & -2 \\ 0 & -2 \end{pmatrix}} \right\}$

(i) Throw away all the redundant vectors

(ii) Express those redundant vectors as linear combination of the remaining (non redundant) vectors.

Ans (a)

$$c_1\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} + c_2\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} + c_3\begin{pmatrix} -1 & -2 \\ -3 & -5 \end{pmatrix} + c_4\begin{pmatrix} -1 & -2 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 0 \\ 2 & 2 & -2 & -2 & 0 \\ 1 & 2 & -3 & 0 & 0 \\ 3 & 4 & -5 & -2 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$c_3 = \alpha$ $\quad$ $c_4 = \beta$

$c_1 = -\alpha + 2\beta$ $\quad$ $c_2 = 2\alpha - \beta$

ie. $(-\alpha + 2\beta)A + (2\alpha - \beta)B + \alpha C + \beta D = 0$

$$\alpha C + \beta D = (\alpha - 2\beta) A + (-2\alpha + \beta) B$$

$$\alpha = 1,\ \beta = 0 \Longrightarrow C = A - 2B$$

$$\begin{pmatrix} -1 & -2 \\ -3 & -5 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} - 2 \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

$$\alpha = 0,\ \beta = 1 \Longrightarrow D = -2A + B$$

$$\begin{pmatrix} -1 & -2 \\ 0 & -2 \end{pmatrix} = -2 \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$