MA 351: Introduction to Linear Algebra and Its Applications
Spring 2019, Test One

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- This test booklet has FIVE QUESTIONS, totaling 100 points for the whole test. You have 75 minutes to do this test. **Plan your time well. Read the questions carefully.**
- This test is **closed book, with no electronic device**.
- In order to get full credits, you need to give **correct** and **simplified** answers and explain in a **comprehensible way** how you arrive at them.

Name: Answer Key (Major:)

Question	Score
1.(20 pts)	
2.(20 pts)	
3.(20 pts)	
4.(20 pts)	
5.(20 pts)	
Total (100 pts)	

1. Let $A = \begin{array}{c} \begin{matrix} u_1 & u_2 & u_3 & u_4 \end{matrix} \\ \begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & -1 & 1 & 1 \\ 2 & 1 & 5 & 1 \end{pmatrix} \end{array}$.

   (a) Find a basis for Col(A). Write the vector(s) you throw away as a linear combination of the basis vectors.

   (b) Find a basis for Null(A).

Find c_1, c_2, c_3, c_4 such that

$$c_1 u_1 + c_2 u_2 + c_3 u_3 + c_4 u_4 = 0$$

(a)

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 2 & 1 & 5 & 1 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & -3 & 3 & 3 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 3 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\underbrace{\quad\quad}_{\text{pivots}} \quad \underbrace{\quad\quad}_{\text{free}} \Longrightarrow u_3, u_4 \text{ can be thrown away}$$

$\Longrightarrow \{u_1, u_2\}$ basis for Col(A)

$c_3 = \alpha, \; c_4 = \beta, \quad c_1 = -3\alpha - \beta, \; c_2 = \alpha + \beta$

Hence

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$$(-3\alpha - \beta)u_1 + (\alpha + \beta)u_2 + \alpha u_3 + \beta u_4 = 0$$

$\alpha = 1, \beta = 0 \Longrightarrow \quad u_3 = 3u_1 - u_2$

$\alpha = 0, \beta = 1 \Longrightarrow \quad u_4 = u_1 - u_2$

(b) $\quad AX = 0$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -3\alpha - \beta \\ \alpha + \beta \\ \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

A basis for Null(A) is $\left\{ \begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$

2. Given $A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & -3 & 0 \\ 1 & 0 & 3 \end{pmatrix}$ and b is one of the following vectors:

$$\begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}.$$

Determine with of the above vector(s) is such that $AX = b$ is solvable. (You don't need to solve it.)

Method 1 (Find out the most general condition on b to be solvable.)

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & a \\ -1 & -3 & 0 & b \\ 1 & 0 & 3 & c \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 0 & -1 & 1 & a+b \\ 0 & -2 & 2 & c-a \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 0 & -1 & 1 & a+b \\ 0 & 0 & 0 & c-a-2a-2b \end{array}\right)$$

We need $c-3a-2b=0$, i.e. $\boxed{c=3a+2b}$

Only $\begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ satisfy this condition.

Method 2 Instead of solving

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we can solve them together:

$$[A \mid b_1\ b_2\ b_3\ b_4]$$

$$\rightarrow \left[\begin{array}{ccc|cccc} 1 & 2 & 1 & 3 & 2 & 1 & -1 \\ -1 & -3 & 0 & -4 & -3 & 3 & 2 \\ 1 & 0 & 3 & 1 & 1 & 2 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|cccc} 1 & 2 & 1 & 3 & 2 & 1 & -1 \\ 0 & -1 & 1 & -1 & -1 & 4 & 1 \\ 0 & -2 & 2 & -2 & -3 & 1 & 2 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|cccc} 1 & 2 & 1 & 3 & 2 & 1 & -1 \\ 0 & -1 & 1 & -1 & -1 & 4 & 1 \\ 0 & 0 & 0 & 0 & -1 & -7 & 0 \end{array}\right]$$

only these two are solvable

ie. $\begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}$ & $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$

3. (a) Prove that if $\{X_1, X_2, \ldots, X_n\}$ is linearly independent and Y does not belong to Span $\{X_1, X_2, \ldots, X_n\}$, then $\{X_1, X_2, \ldots, X_n, Y\}$ is also linearly independent.

(b) Give $X_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ and $X_2 = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$. Find an X_3 such that $\{X_1, X_2, X_3\}$ forms a basis for R^3.

(a) Consider

$$c_1 X_1 + c_2 X_2 + \cdots + c_n X_n + c_{n+1} Y = 0$$

If $c_{n+1} \neq 0$, then, solve for Y:

$$Y = \left(-\frac{c_1}{c_{n+1}}\right) X_1 + \left(-\frac{c_2}{c_{n+1}}\right) X_2 + \cdots + \left(-\frac{c_n}{c_{n+1}}\right) X_n$$

i.e. $Y \in \text{Span}\{X_1, \cdots, X_n\} \Rightarrow$ Contradiction

Now $c_{n+1} = 0$, Then

$$c_1 X_1 + c_2 X_2 + \cdots + c_n X_n = 0$$

$\Rightarrow c_1 = c_2 = \cdots = c_n = 0$ (Since $X_1, X_2, \cdots X_n$ are lin. ind.)

Hence $c_1 = c_2 = \cdots = c_n = c_{n+1} = 0$, so that

$\{X_1, X_2, \cdots, X_n, Y\}$ is lin. ind.

(b) Find Y so that $Y \notin \text{Span}\{X_1, X_2\}$.

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Let $Y = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. Try to find c_1, c_2 s.t.

$$c_1 X_1 + c_2 X_2 = Y$$

i.e. $\left(\begin{array}{cc|c} 1 & 2 & a \\ 1 & 3 & b \\ -1 & 0 & c \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 2 & a \\ 0 & 1 & b-a \\ 0 & 2 & c+a \end{array}\right)$

$$\to \left(\begin{array}{cc|c} 1 & 2 & a \\ 0 & 1 & b-a \\ 0 & 0 & c+a-2b+2a \end{array}\right)$$

We need $c - 2b + 3a \neq 0$ in order to have $Y \notin \text{Span}\{X_1, X_2\}$

Choose $c = 1, a = 0, b = 0$, i.e. $Y = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Then $\{X_1, X_2, Y\}$ are lin. ind. and forms a basis for $\mathbb{R}^3$ (Since $\dim(\mathbb{R}^3) = 3$.)

4. Let $M = \left\{ A^{3\times 3} \text{ (i.e. } A \text{ is a } 3\times 3 \text{ matrix)} : A \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$. Find a basis for M.

(Hint: write down the most general form of an arbitrary A as $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$.)

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \begin{aligned} a+b-c &= 0 \\ d+e-f &= 0 \\ g+h-i &= 0 \end{aligned}$$

$\Rightarrow$ b, c, e, f, h, i are free

& $a = -b+c$, $d = -e+f$, $g = -h+i$

$$A = \begin{pmatrix} -b+c & b & c \\ -e+f & e & f \\ -h+i & h & i \end{pmatrix}$$

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$$= b\begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + c\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$+ e\begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + f\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$+ h\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix} + i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

basis vectors.

5. Let $X_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$, $X_2 = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$ and $Y_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $Y_2 = \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix}$.

Let $V = \text{Span}\{X_1, X_2\}$ and $W = \text{Span}\{Y_1, Y_2\}$. Is $V = W$?

If yes, write an arbitrary[1] vector from V as linear combination of the Y_1 and Y_2. Similary, write an arbitrary vector from W as linear combination of the X_1 and X_2.

If no, find a vector from V that does not belong to W or a vector from W that does not belong to V.

① Try to express Y_1, Y_2 as lin. comb. of X_1, X_2.
(If successful, then we have $W \subseteq V$).

$[X_1 \; X_2 \,|\, Y_1 \; Y_2]$

$$\Rightarrow \left[\begin{array}{cc|cc} 2 & 3 & 1 & 5 \\ 0 & 1 & -1 & -1 \\ 1 & 2 & 0 & 2 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 2 & 0 & 2 \\ 0 & 1 & -1 & -1 \\ 2 & 3 & 1 & 5 \end{array}\right]$$

$$\to \left[\begin{array}{cc|cc} 1 & 2 & 0 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & -1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 2 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Possible: $\boxed{Y_1 = 2X_1 - X_2, \quad Y_2 = 4X_1 - X_2}$

② Similarly, we need to see if X_1, X_2 can be written as lin. comb. of Y_1, Y_2.

(in fact, we can solve for X_1, X_2 in terms of Y_1, Y_2.

[1]Note: in mathematics, the word "arbitrary" means that it must be very general and cannot be any specific example (of your choice).

$X_1 = -\frac{1}{2}Y_1 + \frac{1}{2}Y_2, \quad X_2 = -2Y_1 + Y_2.)$

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$$[Y_1 \ Y_2 \mid X_1 \ X_2]$$

$$\to \left[\begin{array}{cc|cc} 1 & 5 & 2 & 3 \\ -1 & -1 & 0 & 1 \\ 0 & 2 & 1 & 2 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 5 & 2 & 3 \\ 0 & 4 & 2 & 4 \\ 0 & 2 & 1 & 2 \end{array}\right]$$

$$\to \left[\begin{array}{cc|cc} 1 & 5 & 2 & 3 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 5 & 2 & 3 \\ 0 & 1 & 1/2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cc|cc} 1 & 0 & -1/2 & -2 \\ 0 & 1 & 1/2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Hence $\boxed{X_1 = -\frac{1}{2}Y_1 + \frac{1}{2}Y_2 \ \& \ X_2 = -2Y_1 + Y_2}$

3 $\quad \underline{c_1 X_1 + c_2 X_2} = c_1\left(-\frac{1}{2}Y_1 + \frac{1}{2}Y_2\right) + c_2\left(-2Y_1 + Y_2\right)$

$$= \underline{\left(-\frac{c_1}{2} - 2c_2\right)Y_1 + \left(\frac{c_1}{2} + c_2\right)Y_2}$$

Similarly,

$$\underline{c_1 Y_1 + c_2 Y_2} = c_1(2X_1 - X_2) + c_2(4X_1 - X_2)$$

$$= \underline{(2c_1 + 4c_2)X_1 + (-c_1 - c_2)X_2}$$

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