MA 351: Introduction to Linear Algebra and Its Applications
Spring 2019, Test Two

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- This test booklet has FIVE QUESTIONS, totaling 100 points for the whole test. You have 75 minutes to do this test. **Plan your time well. Read the questions carefully.**
- This test is **closed book, with no electronic device**.
- In order to get full credits, you need to give **correct** and **simplified** answers and explain in a **comprehensible way** how you arrive at them.

Name: Answer Key (Major:)

Question	Score
1.(20 pts)	
2.(20 pts)	
3.(20 pts)	
4.(20 pts)	
5.(20 pts)	
Total (100 pts)	

1. Find a basis for the following subspace:

$$U = \left\{ \begin{bmatrix} a+3b+c & -2b+2c & 2a+8c \\ -3a+2b-14c & a-3b+7c & 5a+3b+17c \end{bmatrix} : a, b, c \in R \right\}$$

$$a\underbrace{\begin{bmatrix} 1 & 0 & 2 \\ -3 & 1 & 5 \end{bmatrix}}_{u_1} + b\underbrace{\begin{bmatrix} 3 & -2 & 0 \\ 2 & -3 & 3 \end{bmatrix}}_{u_2} + c\underbrace{\begin{bmatrix} 1 & 2 & 8 \\ -14 & 7 & 17 \end{bmatrix}}_{u_3}$$

Test for redundancy:

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & -2 & 2 \\ 2 & 0 & 8 \\ -3 & 2 & -14 \\ 1 & -3 & 7 \\ 5 & 3 & 17 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 \\ 0 & -2 & 2 \\ 0 & -6 & 6 \\ 0 & 11 & -11 \\ 0 & -6 & 6 \\ 0 & -12 & 12 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

free var.

Hence, a basis is

$$\left\{ \underset{(u_1)}{\begin{bmatrix} 1 & 0 & 2 \\ -3 & 1 & 5 \end{bmatrix}}, \underset{(u_2)}{\begin{bmatrix} 3 & -2 & 0 \\ 2 & -3 & 3 \end{bmatrix}} \right\}$$

2. Let T be a linear transformation from R^2 to R^3 such that

$$T\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

(a) Find the matrix representation of T. (Hint: find $T\begin{pmatrix} x \\ y \end{pmatrix}$.)

(b) Is T onto? Either prove it or find a vector in R^3 which does not have a pre-image.

(c) Is T one-to-one? Either prove it or find two different vectors X_1 and X_2 such that $T(X_1) = T(X_2)$.

(a) $$\begin{pmatrix} x \\ y \end{pmatrix} = c_1\begin{pmatrix} 2 \\ 3 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 2 & 1 & x \\ 3 & 2 & y \end{array}\right) \to \left(\begin{array}{cc|c} 2 & 1 & x \\ 1 & 1 & y-x \end{array}\right)$$

$$\to \left(\begin{array}{cc|c} 1 & 1 & y-x \\ 2 & 1 & x \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & y-x \\ 0 & -1 & x-2y+2x \end{array}\right)$$

$$\to \left(\begin{array}{cc|c} 1 & 1 & y-x \\ 0 & 1 & -3x+2y \end{array}\right)$$

$$\to \left(\begin{array}{cc|c} 1 & 0 & 2x-y \\ 0 & 1 & -3x+2y \end{array}\right)$$

Hence $c_1 = 2x-y$, $c_2 = -3x+2y$

Hence

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$$\begin{pmatrix} x \\ y \end{pmatrix} = (2x-y)\begin{pmatrix} 2 \\ 3 \end{pmatrix} + (-3x+2y)\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = (2x-y)\,T\begin{pmatrix} 2 \\ 3 \end{pmatrix} + (-3x+2y)\,T\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= (2x-y)\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + (-3x+2y)\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2x-y \\ -3x+2y \\ -5x+3y \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & -1 \\ -3 & 2 \\ -5 & 3 \end{pmatrix}}_{\text{matrix for } T.}\begin{pmatrix} x \\ y \end{pmatrix}$$

(b) $A = \begin{pmatrix} 2 & -1 \\ -3 & 2 \\ -5 & 3 \end{pmatrix}$

Try to solve: $Ax = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

$$\left(\begin{array}{cc|c} 2 & -1 & a \\ -3 & 2 & b \\ -5 & 3 & c \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & -1/2 & a/2 \\ -3 & 2 & b \\ -5 & 3 & c \end{array}\right)$$

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$$\to \left(\begin{array}{cc|c} 1 & -\frac{1}{2} & a \\ 0 & \frac{1}{2} & b+\frac{3a}{2} \\ 0 & \frac{1}{2} & c+\frac{5a}{2} \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -\frac{1}{2} & a \\ 0 & \frac{1}{2} & b+\frac{3a}{2} \\ 0 & 0 & a-b+c \end{array}\right)$$

Need $a-b+c=0$ to have a solution.

So <u>T is not onto</u>.

Choose $a=0, b=0, c=1$. $(a-b+c\neq 0)$ Then

$$AX=\begin{pmatrix}0\\0\\1\end{pmatrix} \text{ has no solution.}$$

(c) T is <u>one-to-one</u> as A has no free variable.

3. (a) Suppose a community has two industries: coal and electricity. In order to produce 1 unit of coal, it requires 0.2 unit of electricity while to produce 1 unit of electricity, it requires 0.6 unit of coal and 0.1 unit of electricity. Now the external demands are 1 million unit of coal and 5 million unit of electicity. How many units of coal and electricity needs to be produced?

(b) (i) With the same data in (a), except that the external demand for coal increases by 1 unit while the demand for electricity remains the same. How would the production level of coal and electricity be changed?

(ii) With the same data in (a), except that the external demand for electricity increases by 1 unit while the demand for coal remains the same. How would the production level of coal and electricity be changed?

(iii) With the same data in (a), except that the external demand for coal decreases by 2 unit while the demand for electricity increases by 5 units. How would the production level of coal and electricity be changed?

(Hint: this problem can be solved efficiently by using matrix notion. Remember the inverse of a 2×2 matrix?)

(a) $C = 0.6E + 10^6$

$E = 0.2C + 0.1E + 5\times10^6$

$C \quad -0.6E = 10^6$

$-0.2C + 0.9E = 5\times10^6$

$$\begin{pmatrix} 1 & -0.6 \\ -0.2 & 0.9 \end{pmatrix}\begin{pmatrix} C \\ E \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}\times10^6$$

$$\begin{pmatrix} C \\ E \end{pmatrix} = \begin{pmatrix} 1 & -0.6 \\ -0.2 & 0.9 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 5 \end{pmatrix}\times10^6$$

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$$\begin{pmatrix} C \\ E \end{pmatrix} = \frac{\begin{pmatrix} 0.9 & 0.6 \\ 0.2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 5 \end{pmatrix} \times 10^6}{0.9 - 0.12}$$

$$= \frac{10^6}{0.78}\begin{pmatrix} 0.9 & 0.6 \\ 0.2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 5 \end{pmatrix} = \frac{10^6}{0.78}\begin{pmatrix} 3.9 \\ 5.2 \end{pmatrix}$$

$$\left(\frac{3.9}{0.78} = \frac{390}{78} = 5 \,, \quad \frac{5.2}{0.78} = \frac{520}{78} = \frac{40}{6} = \frac{20}{3} \right)$$

$$\boxed{= \begin{pmatrix} 5 \times 10^6 \\ \frac{20}{3} \times 10^6 \end{pmatrix} \begin{matrix} \leftarrow \text{coal production} \\ \leftarrow \text{electricity production} \end{matrix}}$$

(b) $$\begin{pmatrix} C \\ E \end{pmatrix} = \frac{1}{0.78}\begin{pmatrix} 0.9 & 0.6 \\ 0.2 & 1 \end{pmatrix}\begin{pmatrix} \underbrace{C_0}_{\text{original demand}} + \underbrace{C_1}_{\text{change in demand}} \\ E_0 + E_1 \end{pmatrix}$$

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$$= \frac{1}{0.78}\begin{pmatrix} 0.9 & 0.6 \\ 0.2 & 1 \end{pmatrix}\begin{pmatrix} C_0 \\ E_0 \end{pmatrix} \leftarrow \text{original production}$$

$$+ \frac{1}{0.78}\begin{pmatrix} 0.9 & 0.6 \\ 0.2 & 1 \end{pmatrix}\begin{pmatrix} C_1 \\ E_1 \end{pmatrix} \leftarrow \text{change in production}$$

(i) change

$$= \frac{1}{0.78}\begin{pmatrix} 0.9 & 0.6 \\ 0.2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{0.78}\begin{pmatrix} 0.9 \\ 0.2 \end{pmatrix} = \begin{pmatrix} \frac{90}{78} \\ \frac{20}{78} \end{pmatrix} = \begin{pmatrix} \frac{15}{13} \\ \frac{10}{39} \end{pmatrix}$$

(ii) change

$$= \frac{1}{0.78}\begin{pmatrix} 0.9 & 0.6 \\ 0.2 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{0.78}\begin{pmatrix} 0.6 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{60}{78} \\ \frac{1}{0.78} \end{pmatrix} = \begin{pmatrix} \frac{10}{13} \\ \frac{50}{39} \end{pmatrix}$$

(iii) change

$$= \frac{1}{0.78}\begin{pmatrix} 0.9 & 0.6 \\ 0.2 & 1 \end{pmatrix}\begin{pmatrix} -2 \\ 5 \end{pmatrix} = (-2)(i) + 5(ii)$$

$$= (-2)\begin{pmatrix} 15/13 \\ 10/39 \end{pmatrix} + 5\begin{pmatrix} 10/13 \\ 50/39 \end{pmatrix} = \begin{pmatrix} \frac{20}{13} \\ \frac{230}{39} \end{pmatrix}$$

4. (It is knonw that for any *square* matrix A, if there is a B such that $AB = I$, then *automatically* $BA = I$. This question explores this type of property for *non-square* matrices.)

Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & -1 & 3 \end{pmatrix}$.

(a) If possible, find B such that $AB = I$. In addition, is B unique? If not, find two different B's, B_1 and B_2, ($B_1 \neq B_2$) such that $AB_1 = AB_2 = I$.

(b) If possible, find C (which might or might not equal B) such that $CA = I$. In addition, is C unique? If not, find two different C's, C_1 and C_2, ($C_1 \neq C_2$) such that $C_1A = C_2A = I$.

(Note: *I refers to an* identity *matrix which by definition must be a* square *matrix. Finding the correct dimensions of I, B and C are part of the question. The I in (a) and (c) might have different dimensions*)

(a) $\begin{pmatrix} 1 & 1 & 0 \\ 2 & -1 & 3 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\left(\begin{array}{ccc|cc} 1 & 1 & 0 & 1 & 0 \\ 2 & -1 & 3 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|cc} 1 & 1 & 0 & 1 & 0 \\ 0 & -3 & 3 & -2 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|cc} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 2/3 & -1/3 \end{array}\right) \rightarrow \left(\begin{array}{ccc|cc} 1 & 0 & 1 & 1/3 & 1/3 \\ 0 & 1 & -1 & \frac{2}{3} & -\frac{1}{3} \end{array}\right)$$

free var

$a = -e + 1/3$, $\quad b = -f + 1/3$

$c = e + 2/3$, $\quad d = f - 1/3$

$e =$ free , $\quad f =$ free

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$$B_1 = \begin{pmatrix} 1/3 & 1/3 \\ 2/3 & -1/3 \\ 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} -2/3 & -2/3 \\ 5/3 & 2/3 \\ 1 & 1 \end{pmatrix}$$

$(e=0)$ $(f=0)$ $(e=1)$ $(f=1)$

So B is not unique.

(b) $$\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 2 & -1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\left.\begin{aligned} a+2b &= 1 \\ a-b &= 0 \\ 3b &= 0 \end{aligned}\right\} \Rightarrow \text{no solution.}$$

Hence C does not exist.

5. You are given the following available responses:

   (a) has *at least* one solution for every b.

   (b) has no solutions for some vectors b.

   (c) has at most one solution for every vector b.

   (d) has infinitely many solutions for some vector b.

   (e) The given information is contradictory, no such system is possible.

   (f) Using (only) the information given does not permit us to conclude that any of the above assertions is necessarily true.

   Choose from the above *all* the correct responses for the following situations. *No explanation is needed. Correct choices add points while incorrect choices subtract points. The minimum is zero for each part.*

   (1) What can you say about the system $AX = b$ if A is a 7×12 matrix and

      i. the rank of A is 5? Ans: (b) (d)

      ii. the rank of A is 7? Ans: (a) (d)

      iii. the rank of A is 9? Ans: (e)

   (2) What can you say about the system $BX = b$ if B is a 11×7 matrix and

      i. the rank of B is 5? Ans: (b) (d)

      ii. the rank of B is 7? Ans: (b) (c)

      iii. the rank of B is 9? Ans: (e)