MA 351: Introduction to Linear Algebra and Its Applications
Fall 2021, Test Two

Instructor: Yip

- This test booklet has FIVE QUESTIONS, totaling 100 points for the whole test. You have 75 minutes to do this test. **Plan your time well. Read the questions carefully.**
- This test is **closed book, closed note, with no electronic device**.
- In order to get full credits, you need to give **correct** and **simplified** answers and explain in a **comprehensible way** how you arrive at them.

Name: Answer Key (Major:)

Question	Score
1.(20 pts)	
2.(20 pts)	
3.(20 pts)	
4.(20 pts)	
5.(20 pts)	
Total (100 pts)	

1. Fill in blanks: (Show your computations.)

(a)
$$\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \boxed{?}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \boxed{?}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \boxed{?}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

(b)
$$2 + x + 3x^2 = \boxed{?}(1 + x + x^2) + \boxed{?}(2 - x + x^2) + \boxed{?}(x - x^2)$$

(c)
$$\begin{pmatrix} 3 & 0 \\ -1 & -2 \end{pmatrix} = \boxed{?}\begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix} + \boxed{?}\begin{pmatrix} 2 & 0 \\ -1 & -1 \end{pmatrix} + \boxed{?}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

(a)
$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 1 & 0 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & -1 & 1 & -3 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\boxed{\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 0\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + (-1)\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}$$

(b) $2+x+3x^2 = c_1(1+x+x^2)+c_2(2-x+x^2)+c_3(x-x^2)$

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$$\begin{cases} 2 = c_1 + 2c_2 + 0c_3 \\ 1 = c_1 - c_2 + c_3 \\ 3 = c_1 + c_2 - c_3 \end{cases}$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 3 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & -3 & 1 & -1 \\ 0 & -1 & -1 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 3 & -1 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -4 & 4 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\boxed{2+x+3x^2 = 2(1+x+x^2)+0(2-x+x^2)-(x-x^2)}$$

(c)

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$$\left(\begin{array}{ccc|c} 2 & 2 & 1 & 3 \\ -1 & 0 & -1 & 0 \\ -1 & -1 & -1 & -1 \\ 0 & -1 & 1 & -2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & 2 & 1 & 3 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 2 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 1 & -1 & 2 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \boxed{\begin{pmatrix} 3 & 0 \\ -1 & -2 \end{pmatrix} = 1\begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix} + 1\begin{pmatrix} 2 & 0 \\ -1 & -1 \end{pmatrix} - \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}$$

2. Can you find a linear transformation T from $\mathbf{R}^2$ to $\mathbf{R}^2$ that maps the triangle ABC to triangle $\widetilde{A}\widetilde{B}\widetilde{C}$, where

$$A = \begin{pmatrix} -1 \\ -4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad C = \begin{pmatrix} -2 \\ 4 \end{pmatrix},$$

and

$$\widetilde{A} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}, \quad \widetilde{B} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}, \quad \widetilde{C} = \begin{pmatrix} -4 \\ -28 \end{pmatrix}?$$

(You can assume that T maps A to $\widetilde{A}$, B to $\widetilde{B}$, and C to $\widetilde{C}$.) If so, find the explicit formula for this linear transformation.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -1 \\ -4 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \Rightarrow \begin{matrix} -a-4b=4 \\ -c-4d=4 \end{matrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \end{pmatrix} \Rightarrow \begin{matrix} a+3b=-3 \\ c+3d=-1 \end{matrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -2 \\ 4 \end{pmatrix} = \begin{pmatrix} -4 \\ -28 \end{pmatrix} \Rightarrow \begin{matrix} -2a+4b=-4 \\ -2c+4d=-28 \end{matrix}$$

$$\left(\begin{array}{cc|cc} -1 & -4 & 4 & 4 \\ 1 & 3 & -3 & -1 \\ -2 & 4 & -4 & -28 \end{array}\right) \Rightarrow \left(\begin{array}{cc|cc} 1 & 4 & -4 & -4 \\ 1 & 3 & -3 & -1 \\ 1 & -2 & +2 & 14 \end{array}\right)$$

$$\to \left(\begin{array}{cc|cc} 1 & 4 & -4 & -4 \\ 0 & 1 & -1 & -3 \\ 0 & 6 & -6 & -18 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 4 & -4 & -4 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

You can use this blank page. $\rightarrow$ $\left(\begin{array}{cc|cc} 1 & 0 & 0 & 8 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$a = 0, \quad b = -1$

$c = 8, \quad d = -3$

$$\boxed{T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 8 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ 8x - 3y \end{pmatrix}}$$

3. Let $A = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}$. Find a basis and the dimension for the following subspace of 2×2 matrices:

$$\mathcal{M} = \{B : AB = BA\}.$$

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \quad \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}$$

$$\begin{pmatrix} a-c & b-d \\ 2a & 2b \end{pmatrix} = \begin{pmatrix} a+2b & -a \\ c+2d & -c \end{pmatrix}$$

$$a-c = a+2b \Longrightarrow 2b+c=0$$

$$b-d=-a \Longrightarrow a+b-d=0$$

$$2a = c+2d \Longrightarrow 2a-c-2d=0$$

$$2b=-c \Longrightarrow 2b+c=0$$

$$\Longrightarrow \left(\begin{array}{cccc|c} 0 & 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 \\ 2 & 0 & -1 & -2 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 2 & 0 & -1 & -2 & 0 \end{array}\right)$$

$$\Longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & -2 & -1 & 0 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

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$$\Rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 0 \\ 0 & 1 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & -1/2 & -1 & 0 \\ 0 & 1 & 1/2 & 0 & 0 \end{array}\right)$$

$c = \alpha \qquad d = \beta$

$$a = \frac{\alpha}{2} + \beta, \quad b = -\frac{\alpha}{2}$$

$$B = \begin{pmatrix} \frac{\alpha}{2} + \beta & -\frac{\alpha}{2} \\ \alpha & \beta \end{pmatrix} = \alpha \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 \end{pmatrix} + \beta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Basis vectors.

$$M = \mathrm{Span}\left\{ \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}, \quad \underline{\dim(M) = 2}$$

can be chosen as

$$\begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}$$

4. Let $\mathcal{S} = \{A^{2\times 2} : A = A^t\}$ and $\mathcal{K} = \{B^{2\times 2} : B = -B^t\}$. ($\mathcal{S}$ and $\mathcal{K}$ are called the space of symmetric and skew-symmetric matrices.)

   (a) Find a basis and the dimension for $\mathcal{S}$.

   (b) Find a basis and the dimension for $\mathcal{K}$.

   (c) If you collect the basis vectors for $\mathcal{S}$ and $\mathcal{K}$ together, do they form a basis for the space of all 2×2 matrices? If so, write a general 2×2 matrix as a linear combination of these basis vectors.

(a) $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

$b = c$, a, d - free

$$A = \begin{pmatrix} a & b \\ b & d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$\underline{\dim(\mathcal{S}) = 3}$ $\qquad$ Basis vectors,

(b) $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = -\begin{pmatrix} a & c \\ b & d \end{pmatrix}$

$$\left.\begin{aligned} a &= -a \Leftrightarrow a = 0 \\ b &= -c, \quad c = -b \\ d &= -d \Leftrightarrow d = 0 \end{aligned}\right\} \quad B = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} = b\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Basis Vector

$\underline{\dim(\mathcal{K}) = 1}$

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(c)
$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} = c_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c_4 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$c_1 = x, \quad c_3 = w$

$$\begin{aligned} c_2 + c_4 &= y \\ c_2 - c_4 &= z \end{aligned} \implies \begin{aligned} c_2 &= \frac{y+z}{2} \\ c_3 &= \frac{y-z}{2} \end{aligned}$$

Hence

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} = x \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{y+z}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + w \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \frac{y-z}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Yes, the basis vectors for S & K, together, form a basis for the space of 2x2 matrices. (They can span, and they have 4 vectors in total. Since dim of 2x2 matrices is 4. Hence they are automatically linear independent.)

5. Consider the following linear transformation from $\mathbf{R}^4$ to $\mathbf{R}^3$:

$$T(X) = \begin{pmatrix} 1 & 1 & 2 & -3 \\ 1 & 0 & 1 & 1 \\ 1 & 2 & 3 & -7 \end{pmatrix} X.$$

(a) Is T onto?

If yes, for a general $Y = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, find $X = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$ such that $T(X) = Y$.

If no, find an explicit condition on Y such that there is an X such that $T(X) = Y$. Find also an example of Y such that there is no X so that $T(X) = Y$.

(b) Is T one-to-one?

If no, find an Y and $X_1 \neq X_2$ such that $Y = T(X_1)$ and $Y = T(X_2)$.

(a) $\left(\begin{array}{cccc|c} 1 & 1 & 2 & -3 & a \\ 1 & 0 & 1 & 1 & b \\ 1 & 2 & 3 & -7 & c \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 2 & -3 & a \\ 0 & 1 & 1 & -4 & a-b \\ 0 & 1 & 1 & -4 & c-a \end{array}\right)$

$\to \left(\begin{array}{cccc|c} 1 & 1 & 2 & -3 & a \\ 0 & 1 & 1 & -4 & a-b \\ 0 & 0 & 0 & 0 & c+b-2a \end{array}\right)$

zero row.

Hence T is not onto

We need $c+b=2a$ in order for $T(X)=Y$ to be solvable.

eg. $Y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, then there is not X s.t. $T(X) = Y$.

(b) T is not one-to-one since there is a free variable

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e.g. Set $Y = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 2 & -3 & 0 \\ 0 & 1 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & -4 & 0 \end{array}\right)$$

$z = \alpha$ $\quad w = \beta$

$x = -\alpha - \beta$

$y = -\alpha + 4\beta$

$\alpha = 1, \beta = 0, \quad x = -1, \ y = -1, \ z = 1, \ w = 0$

$\alpha = 0, \beta = 1, \quad x = -1, \ y = 4, \ z = 0, \ w = 1$

Then

$$\begin{pmatrix} 1 & 1 & 2 & -3 \\ 1 & 0 & 1 & 1 \\ 1 & 2 & 3 & -7 \end{pmatrix} \underbrace{\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}}_{= X_1} = \begin{pmatrix} 1 & 1 & 2 & -3 \\ 1 & 0 & 1 & 1 \\ 1 & 2 & 3 & -7 \end{pmatrix} \underbrace{\begin{pmatrix} -1 \\ 4 \\ 0 \\ 1 \end{pmatrix}}_{= X_2}$$

$X_1 \neq X_2$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$