MA 351: Introduction to Linear Algebra and Its Applications
Spring 2019, Final Exam

Instructor: Yip

- This test booklet has EIGHT QUESTIONS, totaling 160 points for the whole test. You have 120 minutes to do this test. **Plan your time well. Read the questions carefully.**
- This test is **closed book, with no electronic device**.
- In order to get full credits, you need to give **correct** and **simplified** answers and explain in a **comprehensible way** how you arrive at them.

Name: Answer Key (Major:)

Question	Score
1.(20 pts)	
2.(20 pts)	
3.(20 pts)	
4.(20 pts)	
5.(20 pts)	
6.(20 pts)	
7.(20 pts)	
8.(20 pts)	
Total (160 pts)	

1. You are given the following information about a 3×3 matrix A:

$$A\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}1\\1\\2\end{pmatrix},\quad A\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}3\\4\\6\end{pmatrix},\quad A\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}4\\8\\7\end{pmatrix}.$$

Find $A\begin{pmatrix}5\\1\\-1\end{pmatrix}$ and vector X such that $AX=\begin{pmatrix}7\\1\\9\end{pmatrix}$.

$$A\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}=\begin{bmatrix}1&3&4\\1&4&8\\2&6&7\end{bmatrix}$$

$$A=\begin{bmatrix}1&3&4\\1&4&8\\2&6&7\end{bmatrix}\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}^{-1}$$

$$\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&1&1&0&1&0\\0&0&1&0&0&1\end{array}\right]\rightarrow\left[\begin{array}{ccc|ccc}1&1&0&1&0&-1\\0&1&0&0&1&-1\\0&0&1&0&0&1\end{array}\right]$$

$$\rightarrow\left[\begin{array}{ccc|ccc}1&0&0&1&-1&0\\0&1&0&0&1&-1\\0&0&1&0&0&1\end{array}\right]$$

Hence

$$A=\begin{bmatrix}1&3&4\\1&4&8\\2&6&7\end{bmatrix}\begin{bmatrix}1&-1&0\\0&1&-1\\0&0&1\end{bmatrix}=\begin{bmatrix}1&2&1\\1&3&4\\2&4&1\end{bmatrix}$$

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(1)

$$A\begin{bmatrix}5\\1\\-1\end{bmatrix} = \begin{bmatrix}1 & 2 & 1\\1 & 3 & 4\\2 & 4 & 1\end{bmatrix}\begin{bmatrix}5\\1\\-1\end{bmatrix} = \boxed{\begin{bmatrix}6\\4\\13\end{bmatrix}}$$

(2)

$$AX = \begin{bmatrix}7\\1\\9\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}1 & 2 & 1 & 7\\1 & 3 & 4 & 1\\2 & 4 & 1 & 9\end{array}\right] \to \left[\begin{array}{ccc|c}1 & 2 & 1 & 7\\0 & 1 & 3 & -6\\0 & 0 & -1 & -5\end{array}\right]$$

$$\to \left[\begin{array}{ccc|c}1 & 2 & 1 & 7\\0 & 1 & 3 & -6\\0 & 0 & 1 & 5\end{array}\right] \to \left[\begin{array}{ccc|c}1 & 2 & 0 & 2\\0 & 1 & 0 & -21\\0 & 0 & 1 & 5\end{array}\right]$$

$$\to \left[\begin{array}{ccc|c}1 & 0 & 0 & 44\\0 & 1 & 0 & -21\\0 & 0 & 1 & 5\end{array}\right]$$

$$\boxed{X = \begin{bmatrix}44\\-21\\5\end{bmatrix}}$$

2. Consider the following system which involves a parameter a in the coefficient matrix and also the right hand side:

$$\begin{aligned} x + y - z &= 2 \\ x + 2y + z &= 7 \\ x + y + (a^2 - 5)z &= a \end{aligned}$$

(a) Find all those a such that the system has a unique solution.

(b) Find all those a such that the system has infinitely many solutions and SOLVE the system.

(c) Find all those a such that the system has no solution.

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 1 & 7 \\ 1 & 1 & a^2-5 & a \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & a^2-4 & a-2 \end{array}\right]$$

(a) We need $a^2-4 \neq 0$, i.e. $\boxed{a \neq 2, -2}$

(b) We need $a^2-4=0$ and $a-2=0$, i.e. $\boxed{a=2}$

Then the system becomes

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -3 & -3 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3\alpha-3 \\ 5-2\alpha \\ \alpha \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}},$$ α - free var.

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(c) We need $a^2-4=0$ & $a-2\neq 0$

$\swarrow$ $\downarrow$

$a=2,-2,$ $a\neq 2.$

i.e. $\boxed{a \text{ must be } -2.}$

3. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$. Find a basis and the dimension for each of the following subspaces:

(a) $U = \{B^{(2\times 2)} : AB = BA\}$;

(b) $V = \{C^{(2\times 2)} : AC = 0\}$;

(c) $W = \{D^{(2\times 2)} : DA = 0\}$.

(a) U: $AB = BA$

$$\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$$

$$\begin{pmatrix} a+2c & b+2d \\ 3a+6c & 3b+6d \end{pmatrix} = \begin{pmatrix} a+3b & 2a+6b \\ c+3d & 2c+6d \end{pmatrix}$$

$(1,1) \Rightarrow 2c = 3b$, $(1,2) \Rightarrow 2a+5b = 2d$

$(2,1) \Rightarrow 3a+5c = 3d$, $(2,2) \Rightarrow 3b = 2c$

$$a-d = -\frac{5}{2}b$$
$$a-d = -\frac{5}{3}c$$

$\rightarrow \frac{b}{2} = \frac{c}{3} \Longleftrightarrow 3b = 2c$

Hence, the conditions become: $a-d = -\frac{5}{2}b$

$c = \frac{3b}{2}$

b, d are free variables.

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$$B = \begin{pmatrix} d - \frac{5}{2}b & b \\ \frac{3b}{2} & d \end{pmatrix} = b\begin{pmatrix} -\frac{5}{2} & 1 \\ \frac{3}{2} & 0 \end{pmatrix} + d\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\boxed{\text{Basis} = \left\{ \begin{pmatrix} -5/2 & 1 \\ 3/2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}, \quad \text{Dim}(U) = 2}$$

(b) $AC = 0$

$$\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} x + 2z & y + 2w \\ 3x + 6z & 3y + 6w \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Set z, w to be free variables. Then

$x = -2z$, $y = -2w$

Hence

$$\boxed{C = \left\{ \begin{pmatrix} -2z & -2w \\ z & w \end{pmatrix} = z\begin{pmatrix} -2 & 0 \\ 1 & 0 \end{pmatrix} + w\begin{pmatrix} 0 & -2 \\ 0 & 1 \end{pmatrix} \right\}}$$

Basis vectors, $\text{Dim}(V) = 2$

(c) $W = \{D : DA = 0\}$

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$$\begin{pmatrix} x & y \\ z & w \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} x+3y & 2x+6y \\ z+3w & 2z+6w \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Set y & w to be free variables. Then,

$x = -3y$, $z = -3w$.

Hence

$$\boxed{D = \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} -3y & y \\ -3w & w \end{pmatrix} = y\begin{pmatrix} -3 & 1 \\ 0 & 0 \end{pmatrix} + w\begin{pmatrix} 0 & 0 \\ -3 & 1 \end{pmatrix}}$$

Basis vectors, Dim = 2

4. Consider the matrix $A = \begin{pmatrix} 2 & 1 & a & b \\ 0 & 3 & -1 & c \\ 0 & 0 & 2 & d \\ 0 & 0 & 0 & 3 \end{pmatrix}$.

Find all those value(s) a, b, c and d such that A is diagonalizable AND then diagonalize A for those value(s).

$\lambda = 2, 2, 3, 3$

$\lambda = 2$, We need 2 free variables.

$$(A-2I|0) \to \left(\begin{array}{cccc|c} 0 & 1 & a & b & 0 \\ 0 & 1 & -1 & c & 0 \\ 0 & 0 & 0 & d & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 0 & 1 & a & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 0 & 1 & a & 0 & 0 \\ 0 & 0 & -1-a & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

free pivot needs to be free. pivot

Hence $\boxed{a = -1}$

then $$(A-2I|0) \to \left(\begin{array}{cccc|c} 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$X = \begin{pmatrix} \alpha \\ \beta \\ \beta \\ 0 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \boxed{X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}}$$

$\lambda=3$, We need 2 free variables.

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$$(A-3I\,|\,0) \longrightarrow \left(\begin{array}{cccc|c} -1 & 1 & -1 & b & 0 \\ 0 & 0 & -1 & c & 0 \\ 0 & 0 & -1 & d & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & -1 & 1 & -b & 0 \\ 0 & 0 & 1 & -c & 0 \\ 0 & 0 & 1 & -d & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & -1 & 1 & -b & 0 \\ 0 & 0 & 1 & -c & 0 \\ 0 & 0 & 0 & c-d & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & -1 & 0 & c-b & 0 \\ 0 & 0 & 1 & -c & 0 \\ 0 & 0 & 0 & c-d & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

pivot free pivot

$c-d$ need to be zero.

i.e. $d=c$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & -1 & 0 & c-b & 0 \\ 0 & 0 & 1 & -c & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

X_3 X_4

$$X = \begin{pmatrix} \alpha-(c-b)\beta \\ \alpha \\ c\beta \\ \beta \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} b-c \\ 0 \\ c \\ 1 \end{pmatrix}$$

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$$A = QDQ^{-1}$$

$$= \boxed{\begin{bmatrix} 1 & 0 & 1 & b-c \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & & & \\ & 2 & & \\ & & 3 & \\ & & & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & b-c \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1}}$$

5. Let $A = \begin{pmatrix} 1 & -3 \\ 1 & 1 \end{pmatrix}$. Find A^{2019}. Simplify your answer as much as possible.

$$\det(A-\lambda I) = \det\begin{pmatrix} 1-\lambda & -3 \\ 1 & 1-\lambda \end{pmatrix} = (\lambda-1)^2+3=0$$

$$\lambda = 1 \pm \sqrt{3}i$$

For $\lambda_1 = 1+\sqrt{3}i$, $\left(A-(1+\sqrt{3}i)I \,\middle|\, 0\right)$

$$\rightarrow \left(\begin{array}{cc|c} -\sqrt{3}i & -3 & 0 \\ 1 & -\sqrt{3}i & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & -\sqrt{3}i & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$X_1 = \begin{pmatrix} \sqrt{3}i \\ 1 \end{pmatrix}$$

For $\lambda_2 = \bar{\lambda}_1 = 1-\sqrt{3}i$, $X_2 = \bar{X}_1 = \begin{pmatrix} -\sqrt{3}i \\ 1 \end{pmatrix}$

$$A = \begin{pmatrix} \sqrt{3}i & -\sqrt{3}i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1+\sqrt{3}i & 0 \\ 0 & 1-\sqrt{3}i \end{pmatrix}\begin{pmatrix} \sqrt{3}i & -\sqrt{3}i \\ 1 & 1 \end{pmatrix}^{-1}$$

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$$A^{2019} = \begin{pmatrix} \sqrt{3}i & -\sqrt{3}i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (1+\sqrt{3}i)^{2019} & \\ & (1-\sqrt{3}i)^{2019} \end{pmatrix} \begin{pmatrix} \sqrt{3}i & -\sqrt{3}i \\ 1 & 1 \end{pmatrix}^{-1}$$

$$1+\sqrt{3}i = 2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$(1+\sqrt{3}i)^{2019} = 2^{2019}\left(\cos\frac{2019}{3}\pi + i\sin\frac{2019}{3}\pi\right)$$

$$= 2^{2019}\left(\cos(673\pi) + i\sin(673\pi)\right)$$

$$= 2^{2019}\left(\cos\pi + i\sin\pi\right) = -2^{2019}$$

$$(1-\sqrt{3}i)^{2019} = 2^{2019}\left(\cos\frac{2019}{3}\pi - i\sin\frac{2019}{3}\pi\right)$$

$$= 2^{2019}\left(\cos 673\pi - i\sin 673\pi\right)$$

$$= -2^{2019}$$

Hence

$$\boxed{A = Q\begin{pmatrix} -2^{2019} & 0 \\ 0 & -2^{2019} \end{pmatrix}Q^{-1} = -2^{2019}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

6. Prove that, for any positive integer n, we have

$$\begin{pmatrix} 4 & -3 \\ 1 & 0 \end{pmatrix}^n = \frac{3^n - 1}{2} \begin{pmatrix} 4 & -3 \\ 1 & 0 \end{pmatrix} + \frac{3 - 3^n}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$A = \begin{pmatrix} 4 & -3 \\ 1 & 0 \end{pmatrix}, \quad \det(A - \lambda I) = \det \begin{pmatrix} 4-\lambda & -3 \\ 1 & -\lambda \end{pmatrix}$$

$$= \lambda^2 - 4\lambda + 3 = 0 \Rightarrow \boxed{\lambda = 1, 3}$$

$$\lambda_1 = 1, \quad (A - I \mid 0) \to \left(\begin{array}{cc|c} 3 & -3 & 0 \\ 1 & -1 & 0 \end{array}\right), \quad X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 3, \quad (A - 3I \mid 0) \to \left(\begin{array}{cc|c} 1 & -3 & 0 \\ 1 & -3 & 0 \end{array}\right), \quad X_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Hence

$$A = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & 3 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}^{-1}$$

$$L.H.S. = A^n = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3^n \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3^n \end{pmatrix} \begin{pmatrix} 1 & -3 \\ -1 & 1 \end{pmatrix} \Big/ -2$$

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$$= \begin{pmatrix} 1 & 3^{n+1} \\ 1 & 3^{n} \end{pmatrix} \begin{pmatrix} -1 & 3 \\ 1 & -1 \end{pmatrix} \Big/ 2$$

$$= \begin{pmatrix} 3^{n+1}-1 & 3-3^{n+1} \\ 3^{n}-1 & 3-3^{n} \end{pmatrix} \Big/ 2$$

$$R.H.S. = \frac{3^{n}-1}{2} \begin{pmatrix} 4 & -3 \\ 1 & 0 \end{pmatrix} + \frac{3-3^{n}}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

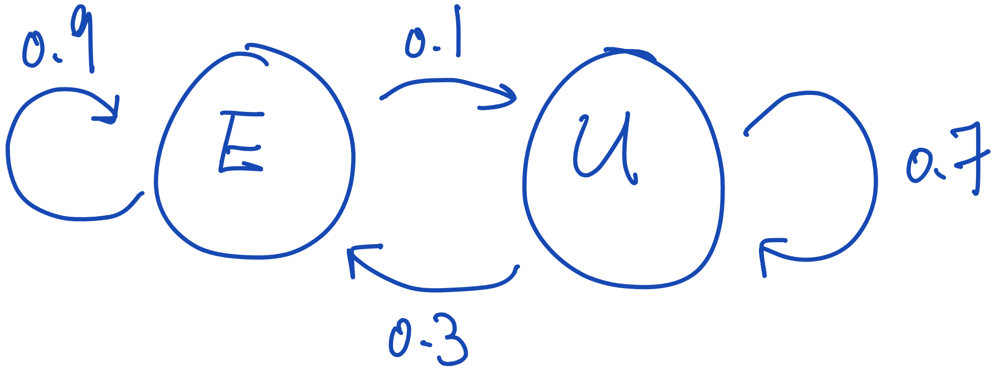
$$= \frac{1}{2} \begin{pmatrix} 4(3^{n})-4+3-3^{n} & -3^{n+1}+3 \\ 3^{n}-1 & 3-3^{n} \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 3^{n+1}-1 & 3-3^{n+1} \\ 3^{n}-1 & 3-3^{n} \end{pmatrix}$$

7. Consider the state of "employable people" is classified as Employed and Unemployed. If a person is employed this year, he/she will be employed next year with 90% chance (and unemployed with 10% chance). If a person is unemployed this year, he/she will be employed next year with 30% chance (and unemployed with 70% chance).

   Let a_n and b_n be the population of employed and unemployed people at the n-th year. Suppose a_1 and b_1 equal 87 and 3 millions, respectively. Find a_2, b_2 and a_n and b_n. What are the limiting values of a_n and b_n as n is very very large?

   (Remark: this model is of course too simplistic as the state of the *following* year only depends on the state in the *current* year. This is the Markov assumption. "In reality", of course if a person is employed continuously for many years, he/she will have a much higher chance of staying in the employed state.)



$$a_{n+1} = 0.9\,a_n + 0.3\,b_n$$
$$b_{n+1} = 0.1\,b_n + 0.7\,b_n$$

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = \begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix}$$

$$\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix} \begin{pmatrix} 87 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 79.2 \\ 10.8 \end{pmatrix} \text{million}$$

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$$\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix}^{n-1} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$$

$$\det(A - \lambda I) = \det\begin{pmatrix} 0.9-\lambda & 0.3 \\ 0.1 & 0.7-\lambda \end{pmatrix}$$

$$= (\lambda - 0.9)(\lambda - 0.7) - 0.03$$

$$= \lambda^2 - 1.6\lambda + 0.63 - 0.03$$

$$= \lambda^2 - 1.6\lambda + 0.6$$

$$= (\lambda - 1)(\lambda - 0.6)$$

$$\lambda = 1 \Rightarrow (A - I \mid 0) = \left(\begin{array}{cc|c} -0.1 & 0.3 & 0 \\ 0.1 & -0.3 & 0 \end{array}\right), \quad X_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\lambda = 0.6 \Rightarrow (A - 0.6I \mid 0) = \left(\begin{array}{cc|c} 0.3 & 0.3 & 0 \\ 0.1 & 0.1 & 0 \end{array}\right), \quad X_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

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$$A = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0.6 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}^{-1}$$

$$A^n = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0.6^n \end{pmatrix} \begin{pmatrix} -1 & -1 \\ -1 & 3 \end{pmatrix} / (-4)$$

$$= \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0.6^n \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} / 4$$

$$= \begin{pmatrix} 3 & 0.6^n \\ 1 & -0.6^n \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} / 4$$

$$= \frac{1}{4} \begin{pmatrix} 3 + 0.6^n & 3 - 3(0.6)^n \\ 1 - 0.6^n & 1 + 3(0.6)^n \end{pmatrix}$$

$$\begin{pmatrix} a^n \\ b^n \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 3 + 0.6^{n-1} & 3 - 3(0.6)^{n-1} \\ 1 - 0.6^{n-1} & 1 + 3(0.6)^{n-1} \end{pmatrix} \begin{pmatrix} 87 \\ 3 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} (3+0.6^{n-1})87 + 9 - 9(0.6)^{n-1} \\ 87 - 87(0.6)^{n-1} + 3 + 9(0.6)^{n-1} \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 270 + 78(0.6)^{n-1} \\ 90 - 78(0.6)^{n-1} \end{pmatrix}$$

$$\boxed{\begin{aligned} a_n &= \frac{1}{4}\left(270 + 78(0.6)^{n-1}\right) \\ b_n &= \frac{1}{4}\left(90 - 78(0.6)^{n-1}\right) \end{aligned}}$$

as $n \to \infty$,

$$\boxed{\begin{aligned} a_n &\longrightarrow \frac{270}{4} \\ b_n &\longrightarrow \frac{90}{4} \end{aligned}}$$

Note: $a_n + b_n = 90 = (87+3)$ million.

8. Suppose a certain country is engaging in three industries: agriculture, manufacturing, and service. You are given the following data:

   In order to produce 1 unit of agriculture it requires 0.1 unit of agriculture, 0.2 unit of machinery, and 0.2 unit of service; while to produce 1 unit of machinery, it requires 0.2 unit of agriculture, 0.3 unit of machinery, and 0.3 unit of service; and finally to produce 1 unit of service, it requires 0.1 unit of agriculture, 0.1 unit of machinery, and 0.3 unit of service;

   Now suppose the external demands for machinery and service increase annually by 2 and 3 units while that for agriculture remains unchanged.

   How would the production levels (in terms of units to be produced) of agriculture, machinery and service change annually? *Be as quantitative as possible in your answer.*

   (Hint: you might want to use some "formula" for the solution of linear systems.)

$$A = 0.1A + 0.2M + 0.1S + 0$$

$$M = 0.2A + 0.3M + 0.1S + 2$$

$$S = 0.2A + 0.3M + 0.3S + 3$$

$$\begin{bmatrix} A \\ M \\ S \end{bmatrix} = \begin{bmatrix} 0.1 & 0.2 & 0.1 \\ 0.2 & 0.3 & 0.1 \\ 0.2 & 0.3 & 0.3 \end{bmatrix} \begin{bmatrix} A \\ M \\ S \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 0.9 & -0.2 & -0.1 \\ -0.2 & 0.7 & -0.1 \\ -0.2 & -0.3 & 0.7 \end{bmatrix} \begin{bmatrix} A \\ M \\ S \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$$

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$$A = \frac{\det\begin{bmatrix} 0 & -0.2 & -0.1 \\ 2 & 0.7 & -0.1 \\ 3 & -0.3 & 0.7 \end{bmatrix}}{\det\begin{bmatrix} 0.9 & -0.2 & -0.1 \\ -0.2 & 0.7 & -0.1 \\ -0.2 & -0.3 & 0.7 \end{bmatrix}}$$

$\boxed{\text{Cramer's Rule}}$

$$= 0.9\begin{vmatrix} 0.7 & -0.1 \\ -0.3 & 0.7 \end{vmatrix} + 0.2\begin{vmatrix} -0.2 & -0.1 \\ -0.2 & 0.7 \end{vmatrix} - 0.1\begin{vmatrix} -0.2 & 0.7 \\ -0.2 & -0.3 \end{vmatrix}$$

$$= 0.9(0.49 - 0.03) + 0.2(-0.14 - 0.02) - 0.1(0.06 + 0.14)$$

$$= 0.9(0.46) - 0.2(0.16) - 0.1(0.2)$$

$$= 0.414 - 0.032 - 0.02 = 0.362$$

$$-2\begin{pmatrix} -0.2 & -0.1 \\ -0.3 & 0.7 \end{pmatrix} + 3\begin{pmatrix} -0.2 & -0.1 \\ 0.7 & -0.1 \end{pmatrix}$$

$$= -2(-0.14 - 0.03) + 3(0.02 + 0.07)$$

$$= 2(0.17) + 0.27 = 0.34 + 0.27 = 0.61$$

$$\boxed{\text{change in } A = \frac{610}{362} \text{ (unit per year)}}$$

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$$M = \frac{1}{0.362} \det \begin{bmatrix} 0.9 & 0 & -0.1 \\ -0.2 & 2 & -0.1 \\ -0.2 & 3 & 0.7 \end{bmatrix}$$

$$= \frac{1}{0.362} \Big[\underbrace{2 \begin{vmatrix} 0.9 & -0.1 \\ -0.2 & 0.7 \end{vmatrix} - 3 \begin{vmatrix} 0.9 & -0.1 \\ -0.2 & -0.1 \end{vmatrix}}_{2(0.63-0.02)-3(-0.09-0.02)} \Big]$$

$$= 2(0.61) + 0.33 = 1.22 + 0.33 = \underline{1.55}$$

$$\boxed{M = \frac{1550}{362}}$$

$$S = \frac{1}{0.362} \det \begin{bmatrix} 0.9 & -0.2 & 0 \\ -0.2 & 0.7 & 2 \\ -0.2 & -0.3 & 3 \end{bmatrix}$$

$$= \frac{1}{0.362} \Big[\underbrace{-2 \begin{vmatrix} 0.9 & -0.2 \\ -0.2 & -0.3 \end{vmatrix} + 3 \begin{vmatrix} 0.9 & -0.2 \\ -.2 & 0.7 \end{vmatrix}}_{-2(-0.27-0.04)+3(0.63-0.04)} \Big]$$

$$= 0.62 + 3(0.59)$$

$$= 0.62 + 1.77 = \underline{2.39}$$

$$\boxed{S = \frac{2390}{362}}$$