MA 351: Introduction to Linear Algebra and Its Applications
Fall 2021, Final Exam

Instructor: Yip

- This test booklet has SEVEN QUESTIONS, totaling 140 points for the whole test. You have 120 minutes to do this test. **Plan your time well. Read the questions carefully.**
- This test is **closed book, closed note, with no electronic device**.
- In order to get full credits, you need to give **correct** and **simplified** answers and explain in a **comprehensible way** how you arrive at them.

Name: Answer Key (Major:)

Question	Score
1.(20 pts)	
2.(20 pts)	
3.(20 pts)	
4.(20 pts)	
5.(20 pts)	
6.(20 pts)	
7.(20 pts)	
Total (140 pts)	

1. You are given the following list of vectors:

$$\overset{A}{\begin{pmatrix} 1 & 0 \\ -1 & 3 \end{pmatrix}}, \overset{B}{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}}, \overset{C}{\begin{pmatrix} 2 & 1 \\ -1 & 7 \end{pmatrix}}, \overset{D}{\begin{pmatrix} 1 & 1 \\ 0 & 4 \end{pmatrix}}.$$

(a) Find a basis for the subspace spanned by the above vectors.

(b) Express each of the above vectors as a linear combination of the basis vectors you have just found.

(a) Consider $c_1 A + c_2 B + c_3 C + c_4 D = 0$

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ -1 & 1 & -1 & 0 & 0 \\ 3 & 1 & 7 & 4 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right)$$

$$\to \begin{array}{c} \left(\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \\ \begin{array}{cccc} & & \alpha & \beta \end{array} \end{array}$$

C and D are redundant.

Basis $= \{A, B\}$

(b) $c_1 = -2\alpha - \beta$, $c_2 = -\alpha - \beta$, $c_3 = \alpha$, $c_4 = \beta$

$\alpha = 1, \beta = 0 \Rightarrow \quad -2A - B + C = 0 \Rightarrow C = 2A + B$

$\alpha = 0, \beta = 1 \Rightarrow \quad -A - B + D = 0 \Rightarrow D = A + B$

2. Fill in blanks.

$$\underbrace{\begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{pmatrix}}_{A} \begin{pmatrix} & X & \\ & ? & \\ & & \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & -1 \\ 2 & 2 & 0 \\ 3 & 1 & 0 \end{pmatrix}}_{B} \Longrightarrow X = A^{-1}B$$

$$\begin{pmatrix} & Y & \\ & ? & \\ & & \end{pmatrix} \underbrace{\begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} 1 & 0 & -1 \\ 2 & 2 & 0 \\ 3 & 1 & 0 \end{pmatrix}}_{C} \Longrightarrow Y = CA^{-1}$$

Find A^{-1}:

$$\left(\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 4 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & -2 & 0 \\ 0 & -2 & -3 & 0 & -4 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 2 & 0 \\ 0 & -2 & -3 & 0 & -4 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 2 & 0 \\ 0 & 0 & -5 & -2 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 2 & 0 \\ 0 & 0 & 1 & 2/5 & 0 & -1/5 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & -2/5 & 1 & 1/5 \\ 0 & 1 & 0 & -3/5 & 2 & -1/5 \\ 0 & 0 & 1 & \frac{2}{5} & 0 & -\frac{1}{5} \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{5} & -1 & \frac{2}{5} \\ 0 & 1 & 0 & -\frac{3}{5} & 2 & -\frac{1}{5} \\ 0 & 0 & 1 & \frac{2}{5} & 0 & -\frac{1}{5} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} \frac{1}{5} & -1 & \frac{2}{5} \\ -\frac{3}{5} & 2 & -\frac{1}{5} \\ \frac{2}{5} & 0 & -\frac{1}{5} \end{pmatrix}$$

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$$X = \frac{1}{5}\begin{pmatrix} 1 & -5 & 2 \\ -3 & 10 & -1 \\ 2 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 2 & 2 & 0 \\ 3 & 1 & 0 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} -3 & -8 & -1 \\ 14 & 19 & 3 \\ -1 & -1 & -2 \end{pmatrix}$$

$$Y = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 2 & 0 \\ 3 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -5 & 2 \\ -3 & 10 & -1 \\ 2 & 0 & -1 \end{pmatrix}\frac{1}{5}$$

$$= \frac{1}{5}\begin{pmatrix} -1 & -5 & 3 \\ -4 & 10 & 2 \\ 0 & -5 & 5 \end{pmatrix}$$

3. Let $\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 3$. Find the values of the following determinants.

(a) $\det\begin{pmatrix} g & h & i \\ -a+d & -b+e & -c+f \\ d & e & f \end{pmatrix}$.

(b) $\det\begin{pmatrix} 3a & 3b & 3c \\ 2d+a & 2e+b & 2f+c \\ -g-a & -h-b & -i-c \end{pmatrix}$.

det is a linear function on each row/col, changes sign upon interchanging any two rows/cols.

(a) $= \det\begin{pmatrix} g & h & i \\ -a & -b & -c \\ d & e & f \end{pmatrix} + \det\begin{pmatrix} g & h & i \\ d & e & f \\ d & e & f \end{pmatrix}$ $=0$

$= \det\begin{pmatrix} a & b & c \\ g & h & i \\ d & e & f \end{pmatrix}$

$= -\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = -3$

(b) $= 3\det\begin{pmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ -g-a & -h-b & -i-c \end{pmatrix}$

$= 3\det\begin{pmatrix} a & b & c \\ 2d & 2e & 2f \\ -g-a & -h-b & -i-c \end{pmatrix} + 3\det\begin{pmatrix} a & b & c \\ a & b & c \\ \cdot & \cdot & \cdot \end{pmatrix}$ $=0$

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$$= 6 \det \begin{pmatrix} a & b & c \\ d & e & f \\ -g & -h & -i \end{pmatrix} + 6 \det \underbrace{\begin{pmatrix} a & b & c \\ d & e & f \\ -a & -b & -c \end{pmatrix}}_{} = 0$$

$$= -6 \det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$= -18$$

4. Consider the following 3×4 matrix:

$$B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 1 \\ -1 & 2 & 1 & 3 \end{pmatrix}.$$

(a) Find the dimensions of Col(B), Null(B), and Row(B).

(b) Find a basis for Col(B), Null(B), and Row(B).

(c) Do the basis vectors of Col(B) form a basis for R^3? If not, find some additional vector(s) so that combined together they do form a basis for R^3.

(d) If you combine the basis vectors of Null(B) and Row(B), do they form a basis for R^4? If not, find some additional vector(s) so that combined together they do form a basis for R^4.

(a) $$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ -1 & 2 & 1 & 3 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & +1 & +3 & +3 & 0 \\ 0 & 4 & 4 & 7 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 1 & 3 & 3 & 0 \\ 0 & 0 & -8 & -5 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 1 & 3 & 3 & 0 \\ 0 & 0 & 1 & 5/8 & 0 \end{array}\right)$$

free

$\dim(\mathrm{Col}(A)) = \dim(\mathrm{Row}(A)) = 3, \quad \dim(\mathrm{Null}(A)) = 1$

(b) Basis for $\mathrm{Col}(A) = \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \right\}$

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$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 1 & 3 & 3 & 0 \\ 0 & 0 & 1 & 5/8 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 2 & 0 & 17/8 & 0 \\ 0 & 1 & 0 & 9/8 & 0 \\ 0 & 0 & 1 & 5/8 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1/8 & 0 \\ 0 & 1 & 0 & 9/8 & 0 \\ 0 & 0 & 1 & 5/8 & 0 \end{array}\right)$$

α

Null(A) : $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{\alpha}{8} \\ -\frac{9\alpha}{8} \\ -\frac{5\alpha}{8} \\ \alpha \end{pmatrix} \overset{\alpha=8}{=} \begin{pmatrix} 1 \\ -9 \\ -5 \\ 8 \end{pmatrix}$ Basis

Row(A) : Basis $= \left\{ \begin{array}{l} (1 \quad 0 \quad 0 \quad -1/8), \\ (0 \quad 1 \quad 0 \quad 9/8), \\ (0 \quad 0 \quad 1 \quad 5/8) \end{array} \right\}$

(c) Yes. (Both Col(A) and $\mathbb{R}^3$ has dim = 3. Basis for Col(A) must span $\mathbb{R}^3$.)

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(d)
$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -9 & 0 \\ 0 & 0 & 1 & -5 & 0 \\ -\frac{1}{8} & \frac{9}{8} & \frac{5}{8} & 8 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -9 & 0 \\ 0 & 0 & 1 & -5 & 0 \\ -1 & 9 & 5 & 64 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -9 & 0 \\ 0 & 0 & 1 & -5 & 0 \\ 0 & 9 & 5 & 65 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -9 & 0 \\ 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 5 & 146 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -9 & 0 \\ 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 171 & 0 \end{array}\right)$$

<u>No free var.</u> Yes. Basis of Row and Null

Combined together form a basis for $\mathbb{R}^4$.

5. This problem explores the property of matrices which might not be square.

Let A be some given general matrix.

(a) Suppose there is a matrix B such that $AB = I$. Show that the linear transformation $X \longrightarrow AX$ is onto.

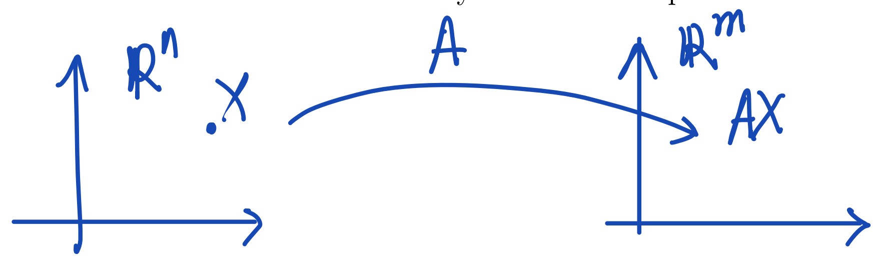
(b) Suppose there is a matrix C such that $CA = I$. Show that the linear transformation $X \longrightarrow AX$ is one-to-one.

Now let $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

(c) If possible, find a matrix B such that $AB = I$. Is your answer unique?

(d) If possible, find a matrix C such that $CA = I$. Is your answer unique?

$A^{m\times n}$:



(a) In order for $X \longrightarrow AX$ to be onto, we need for any Y, there is an X s.t. $AX = Y$, i.e. $AX = Y$ is always solvable for any Y

Let $X = BY$. Then $AX = A(BY) = (AB)Y = Y$

i.e. for any Y, $X = BY$ is a solution

(b) In order for $X \longrightarrow AX$ to be one-to-one, we need if $AX = Y$ has a solution, the solution is unique, i.e. no free variable.

(or if $AX_1 = AX_2$, then $X_1 = X_2$,

or if $AX = 0$, then $X = 0$.)

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If $AX = Y \Rightarrow \overset{I}{\cancel{CA}}X = CY$

$\Rightarrow X = CY$ ← this is the only solution i.e. no free var.

(or if $AX_1 = AX_2 \Rightarrow \cancel{CA}X_1 = \cancel{CA}X_2$

$\Longrightarrow X_1 = X_2$

or if $AX = 0 \Longrightarrow \cancel{CA}X = C0 = 0$

$\Rightarrow X = 0.$)

(c) Let $B = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}$: $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\left(\begin{array}{ccc|cc} 1 & 2 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|cc} 1 & 2 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|cc} 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 \end{array}\right) \to \left(\begin{array}{ccc|cc} 1 & 0 & 1 & -1 & 2 \\ 0 & 1 & 0 & 1 & -1 \end{array}\right)$$

$$a + e = -1 \qquad\qquad b + f = 2$$
$$c = 1 \qquad\qquad d = -1$$

$a = -1 - e$ $\qquad$ $b = 2 - f$

$c = 1$ $\qquad$ $d = -1$

e : free $\qquad$ f = free

Not unique, due to free var.

$$\boxed{\text{Check} \quad \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1-e & 2-f \\ 1 & -1 \\ e & f \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

(d) Not possible as A is 2×3, it must have a free variable. Hence the map $X \longrightarrow AX$ cannot be one-to-one.

(Or try to find C:

$$\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$a+b=1, \quad 2a+b=0, \quad a+b=0$

not possible.

6. Let $A = \begin{pmatrix} 0.3 & 0.5 \\ 0.7 & 0.5 \end{pmatrix}$. Let also $X = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ be an arbitrary vector from R^2.

(a) Find an exact formula for $A^n X$. More precisely, let $\begin{pmatrix} x_n \\ y_n \end{pmatrix} = A^n X$. Find an explicit formula for x_n and y_n.

(b) Find the limiting value of $A^n X$ as $n \longrightarrow \infty$.

(a) $$\det(A-\lambda I) = \det\begin{pmatrix} 0.3-\lambda & 0.5 \\ 0.7 & 0.5-\lambda \end{pmatrix} = (0.3-\lambda)(0.5-\lambda) - 0.35$$

$$= \lambda^2 - 0.8\lambda + 0.15 - 0.35$$

$$= \lambda^2 - 0.8\lambda - 0.2$$

$$= (\lambda-1)(\lambda+0.2)$$

$$\lambda_1 = 1 \Rightarrow (A - I \mid 0) \rightarrow \left(\begin{array}{cc|c} -0.7 & 0.5 & 0 \\ 0.7 & -0.5 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cc|c} 7 & -5 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad X_1 = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$$

$$\lambda_2 = -0.2 \Rightarrow (A + 0.2I \mid 0) \rightarrow \left(\begin{array}{cc|c} 0.5 & 0.5 & 0 \\ 0.7 & 0.7 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad X_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 5 & -1 \\ 7 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -0.2 \end{pmatrix}\begin{pmatrix} 5 & -1 \\ 7 & 1 \end{pmatrix}^{-1}$$

$$A^n = \begin{pmatrix} 5 & -1 \\ 7 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & (-0.2)^n \end{pmatrix}\begin{pmatrix} 5 & -1 \\ 7 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 5 & -1 \\ 7 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (-0.2)^n \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -7 & 5 \end{pmatrix} \frac{1}{12}$$

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$$= \begin{pmatrix} 5 & -(-0.2)^n \\ 7 & (-0.2)^n \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -7 & 5 \end{pmatrix} \frac{1}{12}$$

$$= \frac{1}{12} \begin{pmatrix} 5+7(-0.2)^n & 5-5(-0.2)^n \\ 7-7(-0.2)^n & 7+5(-0.2)^n \end{pmatrix}$$

$$A^n X_0 = \frac{1}{12} \begin{pmatrix} 5+7(-0.2)^n & 5-5(-0.2)^n \\ 7-7(-0.2)^n & 7+5(-0.2)^n \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$= \frac{1}{12} \begin{pmatrix} 5x_0 + 7(-0.2)^n x_0 + 5y_0 - 5(-0.2)^n y_0 \\ 7x_0 - 7(-0.2)^n x_0 + 7y_0 + 5(-0.2)^n y_0 \end{pmatrix}$$

(b) as $n \to +\infty$, $(-0.2)^n \to 0$

$$A^n X_0 \to \frac{1}{12} \begin{pmatrix} 5x_0 + 5y_0 \\ 7x_0 + 7y_0 \end{pmatrix} = (x_0+y_0) \underbrace{\begin{pmatrix} \frac{5}{12} \\ \frac{7}{12} \end{pmatrix}}_{\text{Steady state}}$$

7. Let $P = \frac{1}{1+m^2}\begin{pmatrix} 1 & m \\ m & m^2 \end{pmatrix}$. (It is the matrix that projects an arbitrary vector in R^2 perpendicularly onto the line $y = mx$.)

(a) Show that $P^2 = P$.

(b) Find all the eigenvalues and eigenvectors of P.

Now let $R = 2P - I$. (It is the matrix that reflects an arbitrary vector in R^2 with respect to the line $y = mx$.)

(c) Show that $R^2 = I$.

(d) Find all the eigenvalues and eigenvectors of R.

(a) $$P^2 = \frac{1}{(1+m^2)^2}\begin{pmatrix} 1 & m \\ m & m^2 \end{pmatrix}\begin{pmatrix} 1 & m \\ m & m^2 \end{pmatrix}$$

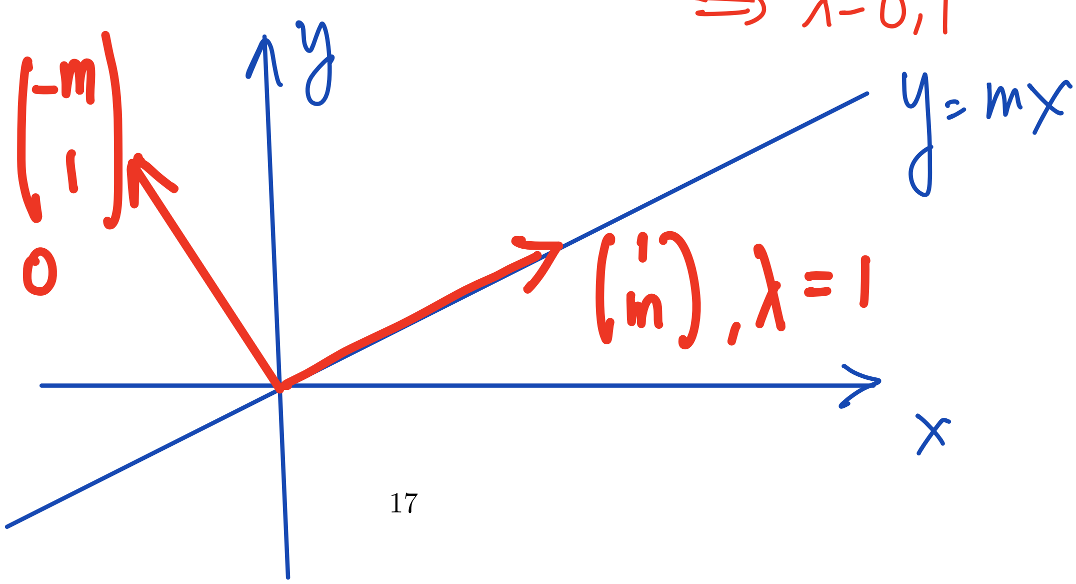
$$= \frac{1}{(1+m^2)^2}\begin{pmatrix} 1+m^2 & m+m^3 \\ m+m^3 & m^2+m^4 \end{pmatrix} = \frac{1}{1+m^2}\begin{pmatrix} 1 & m \\ m & m^2 \end{pmatrix} = P$$

(b) $$PX = \lambda X \Rightarrow P^2X = P(\lambda X) = \lambda PX = \lambda^2 X$$

$$P^2X = PX = \lambda X \Rightarrow \lambda^2 = \lambda$$

$$\Rightarrow \lambda = 0, 1$$

P :



(Or try to find λ and X directly:

$$\det(P-\lambda I) = \det\begin{pmatrix} \frac{1}{1+m^2}-\lambda & \frac{m}{1+m^2} \\ \frac{m}{1+m^2} & \frac{m^2}{1+m^2}-\lambda \end{pmatrix}$$

$$= \left(\frac{1}{1+m^2}-\lambda\right)\left(\frac{m^2}{1+m^2}-\lambda\right) - \frac{m^2}{(1+m^2)^2}$$

$$= \lambda^2 - \lambda + \cancel{\frac{m^2}{(1+m^2)^2}} - \cancel{\frac{m^2}{(1+m^2)^2}}$$

$$= \lambda(\lambda-1) \implies \lambda = 0, 1.)$$

$\lambda_1 = 0$, $\quad PX = 0 \implies \left(\begin{array}{cc|c} 1 & m & 0 \\ m & m^2 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & m & 0 \\ 0 & 0 & 0 \end{array}\right)$

$$\boxed{X_1 = \begin{pmatrix} -m \\ 1 \end{pmatrix}}$$

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$\lambda_2 = 1$, $(P - I)X = 0 \Rightarrow \left(\begin{array}{cc|c} \frac{1}{1+m^2} - 1 & \frac{m}{1+m^2} & 0 \\ \frac{m}{1+m^2} & \frac{m^2}{1+m^2} - 1 & 0 \end{array}\right)$

$$\rightarrow \left(\begin{array}{cc|c} -m^2 & m & 0 \\ m & -1 & 0 \end{array}\right)$$

$$\boxed{X_2 = \begin{pmatrix} 1 \\ m \end{pmatrix}} \qquad \rightarrow \left(\begin{array}{cc|c} m & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

(c) $R = 2P - I$

$$R^2 = (2P - I)^2 = 4P^2 - 4P + I \qquad (P^2 = P)$$

$$= 4P - 4P + I = I$$

(d) $RX = \lambda X \Rightarrow \underbrace{R^2 X}_{= X} = R(\lambda X) = \lambda RX = \lambda^2 X$

$\lambda^2 = 1$

$\lambda = 1, -1$

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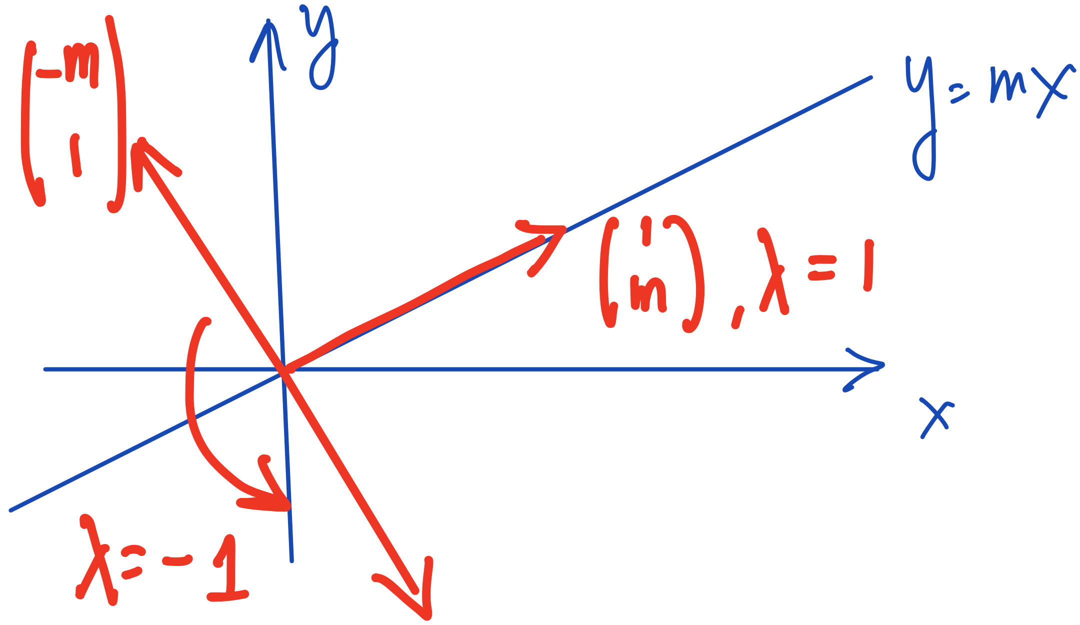
R



(Can also find λ directly:

$$\det(R - \lambda I) = 0 \quad \cdots\cdots \quad)$$

$$R = 2P - I = \frac{1}{1+m^2}\begin{pmatrix} 2 & 2m \\ 2m & 2m^2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2} \\ \frac{2m}{1+m^2} & \frac{m^2-1}{1+m^2} \end{pmatrix}$$

$$\det(R-\lambda I) = \det\begin{pmatrix} \frac{1-m^2}{1+m^2}-\lambda & \frac{2m}{1+m^2} \\ \frac{2m}{1+m^2} & \frac{m^2-1}{1+m^2}-\lambda \end{pmatrix}$$

$$= \left(\frac{1-m^2}{1+m^2}-\lambda\right)\left(\frac{m^2-1}{1+m^2}-\lambda\right) - \frac{4m^2}{(1+m^2)^2}$$

$(a-b)(-a-b)$
$= b^2-a^2$

$$= \lambda^2 - \left(\frac{m^2-1}{1+m^2}\right)^2 - \frac{4m^2}{(1+m^2)^2}$$

$$= \lambda^2 - \frac{m^4-2m^2+1+4m^2}{(1+m^2)^2}$$

$$= \lambda^2 - \frac{m^4+2m^2+1}{(1+m^2)^2}$$

$\leftarrow (a+b)^2$
$= a^2+2ab+b^2$

$$= \lambda^2-1 \Rightarrow \lambda = \pm 1.$$

$\underline{\lambda_1 = 1},$ $(R-I\,|\,0) \rightarrow \left(\begin{array}{cc|c} \frac{-\cancel{2}m^2}{\cancel{1+m^2}} & \frac{\cancel{2}m}{\cancel{1+m^2}} & 0 \\ \frac{\cancel{2}m}{\cancel{1+m^2}} & \frac{-\cancel{2}}{\cancel{1+m^2}} & 0 \end{array}\right)$

$$\rightarrow \left(\begin{array}{cc|c} \cancel{-m} & \cancel{1} & 0 \\ m & -1 & 0 \end{array}\right)$$

$$\boxed{X_1 = \begin{pmatrix} 1 \\ m \end{pmatrix}}$$

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$$\lambda_2 = -1, \quad \left(R + I \,\middle|\, 0\right) \to \left(\begin{array}{cc|c} \frac{\cancel{2}}{\cancel{1+m^2}} & \frac{\cancel{2}m}{\cancel{1+m^2}} & 0 \\ \frac{\cancel{2}m}{\cancel{1+m^2}} & \frac{\cancel{2}m^2}{\cancel{1+m^2}} & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cc|c} 1 & m & 0 \\ \cancel{m} & \cancel{m^2} & \cancel{0} \end{array}\right)$$

$$\boxed{X_2 = \begin{pmatrix} -m \\ 1 \end{pmatrix}}$$