MA 351 Fall 2024 (Aaron N. K. Yip)

Practice Problems (for Chapter 5)

Penney, Linear Algebra: Ideas and Applications (4th edition)
EXERCISES p.281: 5.10, 5.15, 5.16;
EXERCISES p.285: 5.17, 5.20, 5.21, 5.22
(For these problems, *equilibrium* or *steady state vector* is an eigenvector X with eigenvalue 1: $AX = X$, i.e. no change after multiplying by A. Depending on the context, you might need to choose appropriate value of the free variable α when you find X. For example, the entries of X might to required to sum up to 1, 100%, or some other value.)
EXERCISES p.290: 5.27, 5.28, 5.29(a,b,c,d,f), 5.31, 5.32, 5.33, 5.34, 5.35, 5.36, 5.37, 5.38
Revisit Homework 3, Additional Problem. Relate the Google PageRank Vector X to the eigenvector (with appropriate eigenvalue) of some matrix.

Additional Problem.

1. **Linear independence of eigenvectors.**

   In class, we have shown that the eigenvectors corresponding to *distinct, i.e. different* eigenvalues are automatically linearly independent. This problem extends this property to eigenvectors even if some of the eigenvalues are *repeating*. The proof is basically the same but with some twist at the very last step.

   Just to be concrete, let X_1, X_2, X_3 be eigenvectors of A with eigenvalue λ_1, Y_1, Y_2, Y_3 be eigenvectors of A with eigenvalue λ_2, and $\lambda_1 \neq \lambda_2$. You can assume that X_1, X_2, X_3 are linearly independent by themselves and Y_1, Y_2, Y_3 are also linearly independent by themselves – you can assume the last properties by the method you find the X_i's and Y_i's which are nothing but basis of $\mathrm{Null}(A - \lambda_1 I)$ and $\mathrm{Null}(A - \lambda_2 I)$.

   Show that $X_1, X_2, X_3, Y_1, Y_2, Y_3$ are linearly independent by the following proof by contradiction.

   "Suppose" $X_1, X_2, X_3, Y_1, Y_2, Y_3$ are *linearly dependent*. Then one of the following cases must be true.

   (a) $X_1, X_2, X_3, Y_1, Y_2, Y_3$ are linearly dependent but X_1, X_2, X_3, Y_1, Y_2 are linearly independent; or

   (b) X_1, X_2, X_3, Y_1, Y_2 are linearly dependent but X_1, X_2, X_3, Y_1 are linearly independent; or

   (c) X_1, X_2, X_3, Y_1 are linearly dependent but X_1, X_2, X_3 are linearly independent; or

   (d) X_1, X_2, X_3 are linearly dependent but X_1, X_2 are linearly independent; or

(e) X_1, X_2 are linearly dependent but X_1 is linearly independent; or

(f) X_1 is linearly dependent.

Show that none of the above cases is possible and hence the original assumption that X_1, X_2, X_3, Y_1, Y_2 are linearly dependent must be wrong.

(The above idea works for any other cases with as many repeating and as many distinct eigenvalues as possible.)

Hint for (a). (The other parts are the same.) Consider

$$c_1X_1 + c_2X_2 + c_3X_3 + c_4Y_1 + c_5Y_2 + c_6Y_3 = 0. \qquad (*)$$

Multiply $(*)$ by λ_2, multiply $(*)$ by A and subtract, i.e. consider $A(*) - \lambda_2(*)$.

2. **Properties of "similar" matrices.**

A (square) matrix A is said to be *similar to* (square matrix) B, written as $A \sim B$, if there is some invertible matrix P such that $A = PBP^{-1}$. (Analogous to diagonalization, $A = QDQ^{-1}$, similar matrices represent the same linear transformation but in terms of different basis.)

Prove the following lists of properties of similar matrices. (In the following, you can assume that $A \sim B$.)

(a) "Similar" is an *equivalence relation*, i.e.

i. (reflexive) $A \sim A$;

ii. (symmetric) if $A \sim B$, then $B \sim A$;

iii. (transitive) if $A \sim B$ and $B \sim C$, then $A \sim C$.

(b) $\det(A) = \det(B)$. (Hint: use $\det(M^{-1}) = \frac{1}{\det(M)}$.)

(c) $\mathrm{tr}(A) = \mathrm{tr}(B)$. (Hint: use $\mathrm{tr}(MN) = \mathrm{tr}(MN)$.)

(d) $\det(A - \lambda I) = \det(B - \lambda I)$.

(e) The eigenvectors of A can be "naturally" linked to eigenvectors of B, i.e. if $AX = \lambda X$ and $BY = \lambda Y$, try to find a "natural" relationship between X and Y.

(The last two properties essentially state that similar matrices have the same set of eigenvalues and eigenvectors, even counting algebraic and geometric multiplicities.)