Simplex Method [V] p. 20

max $\zeta = 3x_1 + 2x_2$

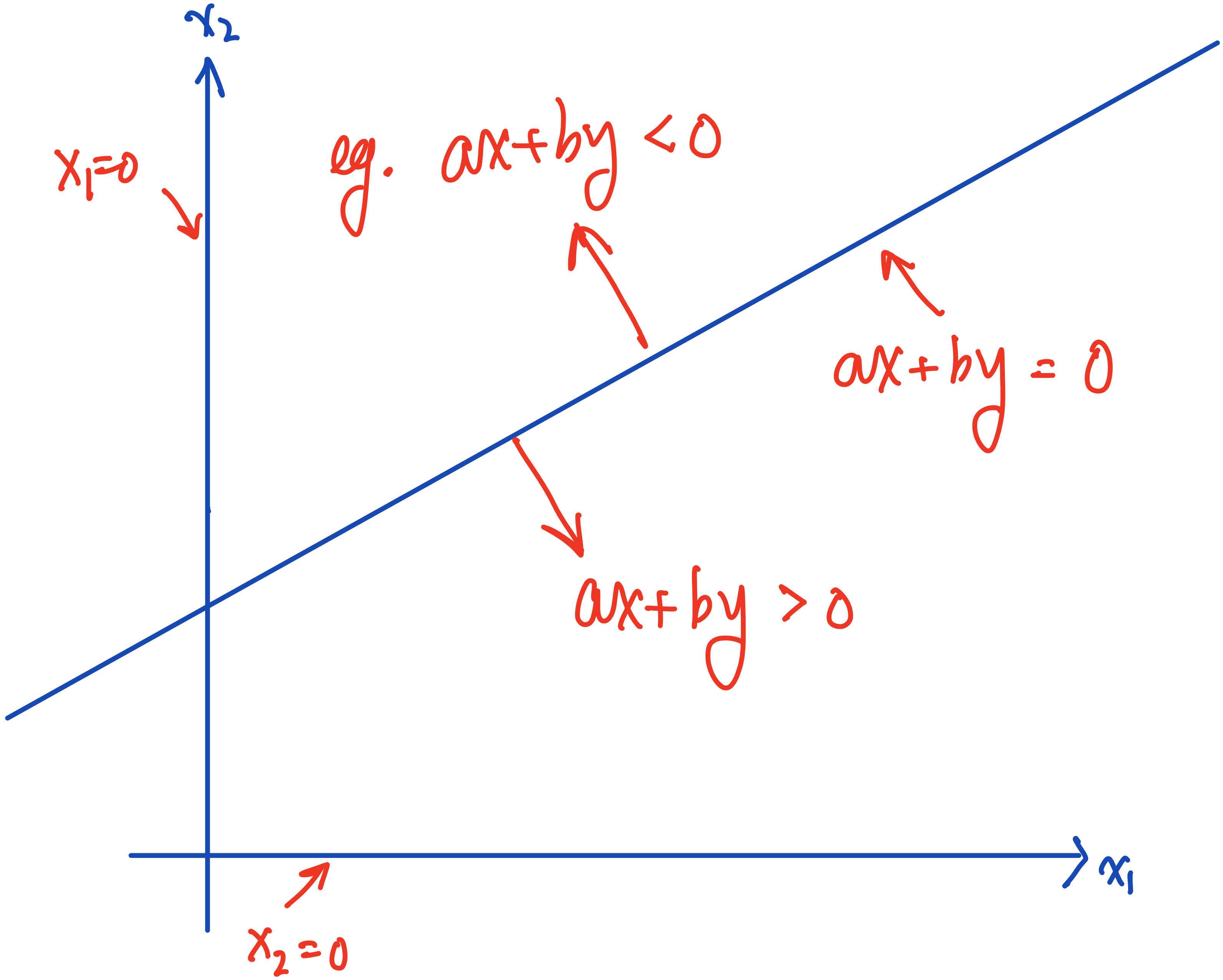
subject to

$$-x_1 + 3x_2 \leq 12$$
$$x_1 + x_2 \leq 8$$
$$2x_1 - x_2 \leq 10$$
$$x_1, x_2 \geq 0$$

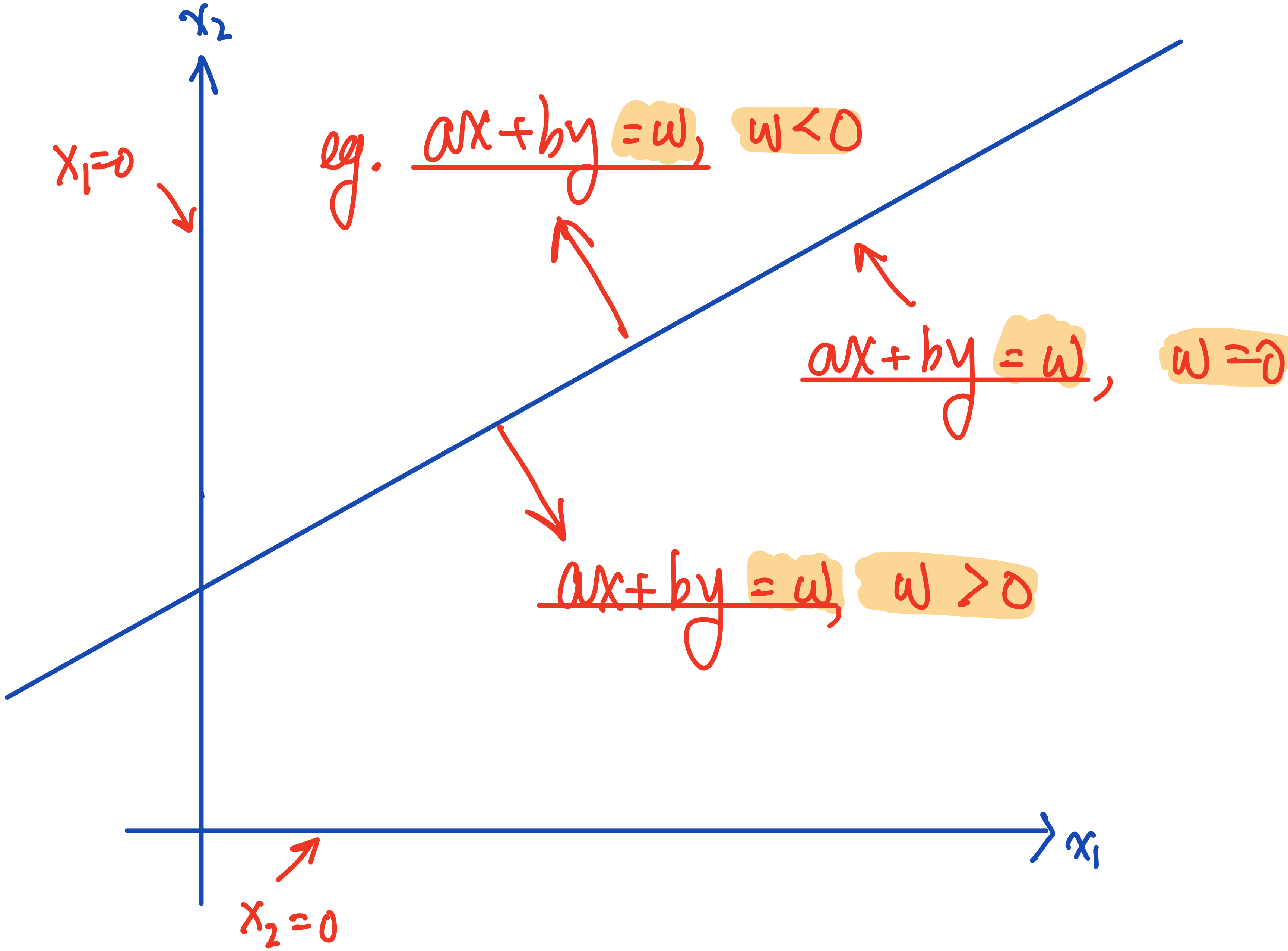
Slack variable form:

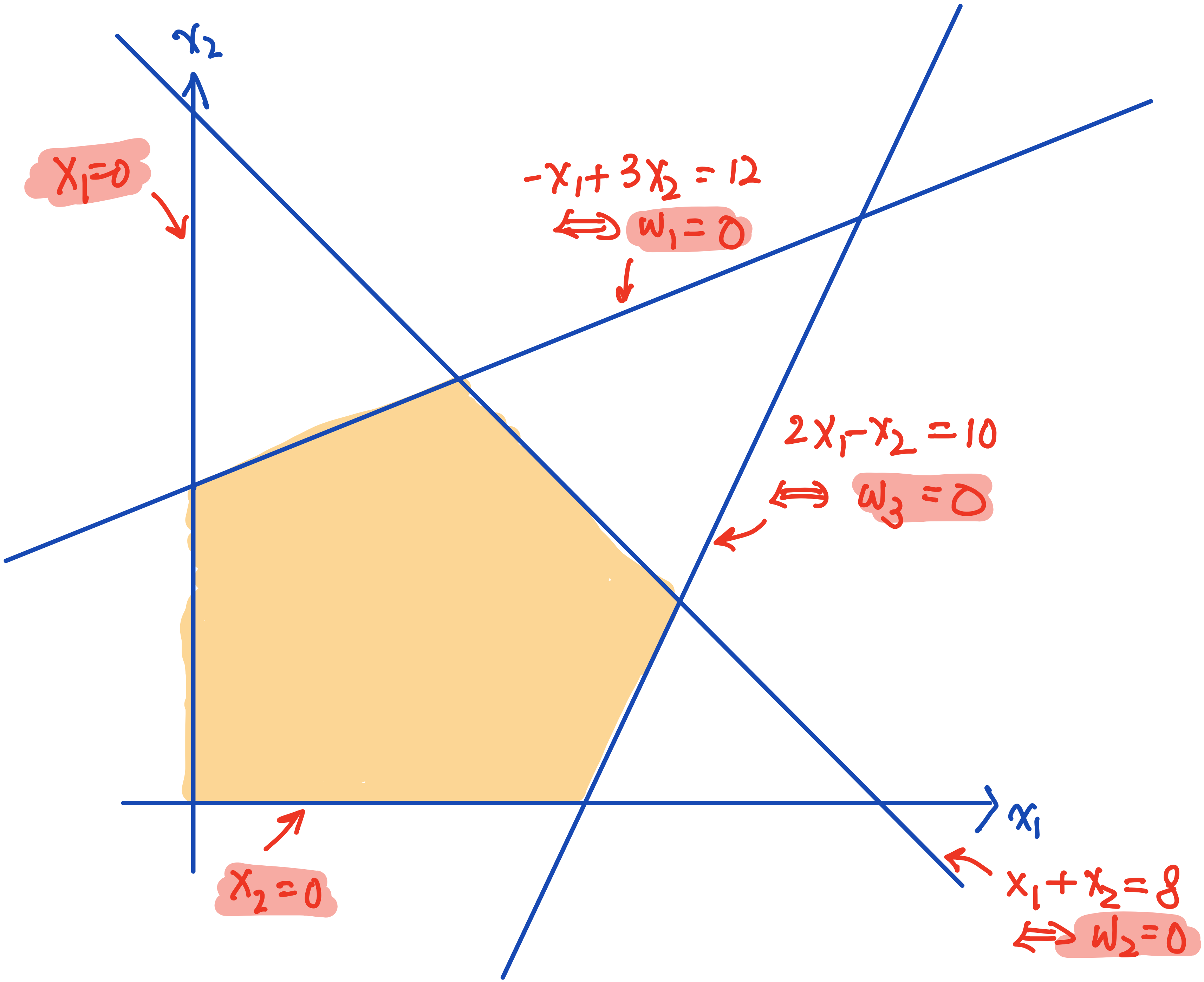
$$w_1 = 12 + x_1 - 3x_2$$
$$w_2 = 8 - x_1 - x_2$$
$$w_3 = 10 - 2x_1 + x_2$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

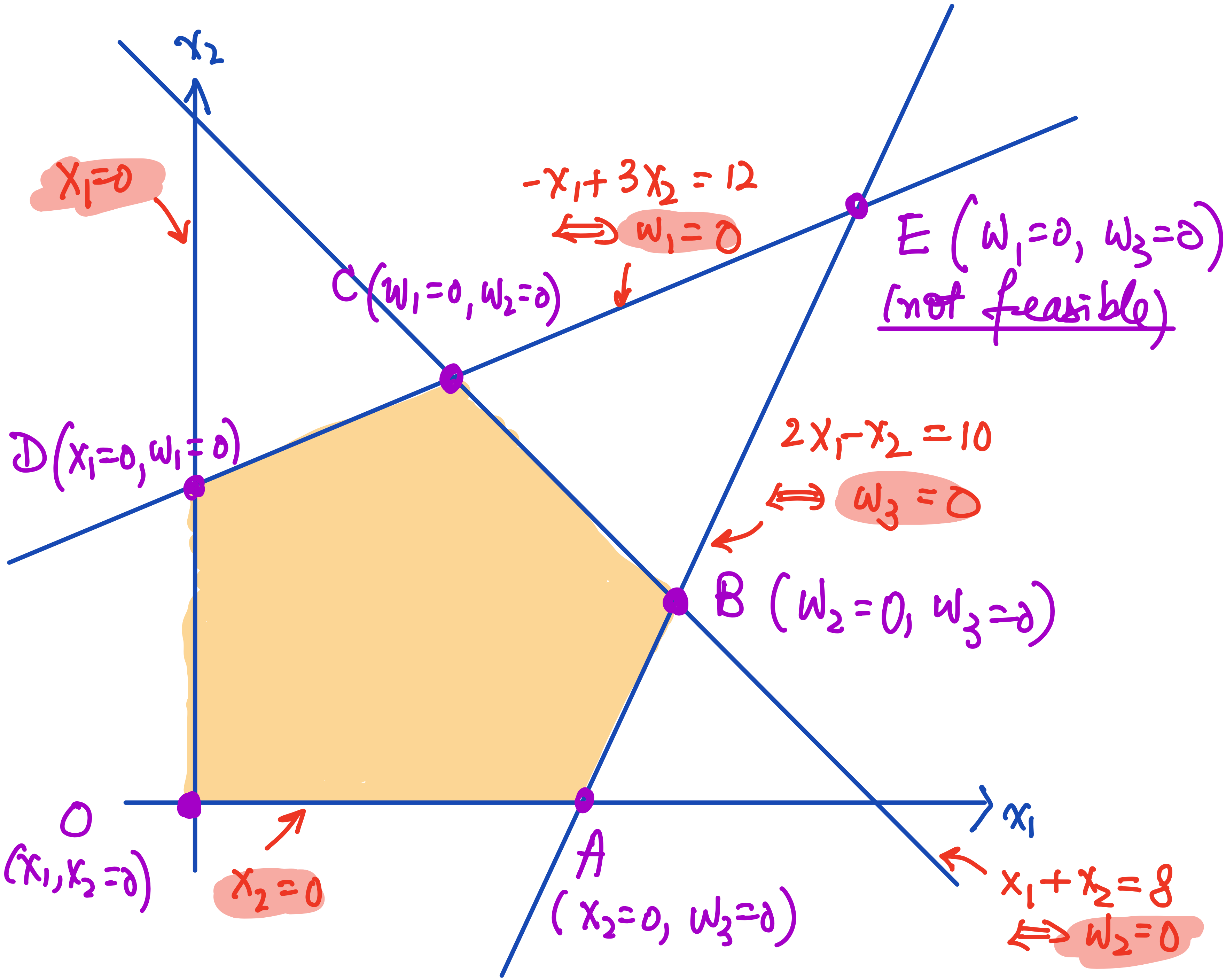


x_2
$x_1 = 0$
eg. $ax+by<0$
$ax+by=0$
$ax+by>0$
$x_2 = 0$
x_1


x_2
$x_1=0$
eg. $ax+by=\omega$, $\omega<0$
$ax+by=\omega$, $\omega=0$
$ax+by=\omega$, $\omega>0$
x_1
$x_2=0$



x_2
x_1
$x_1=0$
$-x_1+3x_2=12$
$\Leftrightarrow w_1=0$
$2x_1-x_2=10$
$\Leftrightarrow w_3=0$
$x_2=0$
$x_1+x_2=8$
$\Leftrightarrow w_2=0$



x_2
x_1
$x_1=0$
$x_2=0$
$-x_1+3x_2=12$
$\Longleftrightarrow w_1=0$
$2x_1-x_2=10$
$\Longleftrightarrow w_3=0$
$x_1+x_2=8$
$\Longleftrightarrow w_2=0$
O $(x_1, x_2=0)$
A $(x_2=0, w_3=0)$
B $(w_2=0, w_3=0)$
C $(w_1=0, w_2=0)$
D $(x_1=0, w_1=0)$
E $(w_1=0, w_3=0)$
(not feasible)

(0) $\max \quad \zeta = 3x_1 + 2x_2$

subject to
$$w_1 = 12 + x_1 - 3x_2$$
$$w_2 = 8 - x_1 - x_2$$
$$w_3 = 10 - 2x_1 + x_2$$
$$x_1,\ x_2,\ x_3,\ w_1,\ w_2,\ w_3 \geq 0$$

x_1, x_2, x_3 on the RHS — non-basic variables

w_1, w_2, w_3 on the LHS ← basic variable

Idea of Simplex Method:

(1) set all non-basic var. to zero — correspond to a vertex

(2) choose a new set of (non)basic vars, to improve ζ.

((1) & (2) is the same as moving from vertex to vertex.)

I Initial Step/Guess

$$\max \zeta = 3x_1 + 2x_2$$

subject to

$$w_1 = 12 + x_1 - 3x_2$$
$$w_2 = 8 - x_1 - x_2$$
$$w_3 = 10 - 2x_1 + x_2$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

Set $x_1, x_2 = 0$ $(0) \Rightarrow \zeta = 0$

① Improve (0) ($x_1=0,\ x_2=0$)

increase x_1, keep $x_2=0$
(max. coeff. rule)

$$\max\ \zeta = 3x_1 + 2x_2$$

subject to
$$\begin{aligned} w_1 &= 12 + x_1 - 3x_2 \\ w_2 &= 8 - x_1 - x_2 \\ w_3 &= 10 - 2x_1 + x_2 \end{aligned}$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

w_1: x_1 can be as large as possible

w_2: $8 - x_1 \geq 0 \implies x_1 \leq 8$

w_3: $10 - 2x_1 \geq 0 \implies x_1 \leq 5$

Set $x_1 = 5$, $\Rightarrow$ $w_3 = 0 \implies A = (x_2=0, w_3=0)$

II $A\ (x_2 = 0,\ w_3 = 0)$

Interchange x_1 and w_3 with w_3 leaving basic and x_1 entering basic

$$\zeta = 3x_1 + 2x_2$$

$$w_1 = 12 + x_1 - 3x_2$$

$$w_2 = 8 - x_1 - x_2$$

$$w_3 = 10 - 2x_1 + x_2$$

$$\boxed{x_1 = 5 - \frac{w_3}{2} + \frac{1}{2}x_2}$$

$$w_1 = 12 + 5 - \frac{w_3}{2} + \frac{1}{2}x_2 - 3x_2 = 17 - \frac{w_3}{2} - \frac{5x_2}{2}$$

$$w_2 = 8 - 5 + \frac{w_3}{2} - \frac{1}{2}x_2 - x_2 = 3 + \frac{w_3}{2} - \frac{3x_2}{2}$$

$$\zeta = 3\left(5 - \frac{w_3}{2} + \frac{1}{2}x_2\right) + 2x_2 = 15 - \frac{3w_3}{2} + \frac{7}{2}x_2$$

(II) $A\ (x_2 = 0,\ w_3 = 0)$

Interchange x_1 and w_3 with w_3 leaving basic and x_1 entering basic

$$\max\ \zeta = 15 - \frac{3}{2} w_3 + \frac{7}{2} x_2$$

subject to

$$x_1 = 5 - \frac{1}{2} w_3 + \frac{x_2}{2}$$

$$w_1 = 17 - \frac{1}{2} w_3 - \frac{5}{2} x_2$$

$$\underbrace{w_3}_{\text{basic var.}} = \underbrace{3 + \frac{1}{2} w_3 - \frac{3}{2} x_2}_{\text{non-basic var.}}$$

III Improve: A ($x_2=0,\ w_3=0$)

$$\max\ \zeta = 15 - \frac{3}{2}w_3 + \frac{7}{2}x_2$$

increase x_2, keeping $w_3=0$

subject to
$$x_1 = 5 - \frac{1}{2}w_3 + \frac{x_2}{2}$$
$$w_1 = 17 - \frac{1}{2}w_3 - \frac{5}{2}x_2$$
$$w_2 = 3 + \frac{1}{2}w_3 - \frac{3}{2}x_2$$

basic var. non-basic var.

x_1: x_2 can be as large as possible

w_1: $17 - \frac{5}{2}x_2 \geq 0 \Longrightarrow x_2 \leq \frac{34}{5}$

w_2: $3 - \frac{3}{2}x_2 \geq 0 \Longrightarrow x_2 \leq 2$

Set $x_2 = 2 \Longrightarrow w_3=0$ and $w_2=0 \Longrightarrow B$

III $\quad B\,(w_2=0,\ w_3=0)$

Interchange x_2 and w_2 with

w_2 leaving basic and x_2 entering basic.

$$\max \ \zeta = 15 - \frac{3}{2}w_3 + \frac{7}{2}x_2$$

subject to
$$x_1 = 5 - \frac{1}{2}w_3 + \frac{x_2}{2}$$
$$w_1 = 17 - \frac{1}{2}w_3 - \frac{5}{2}x_2$$
$$w_2 = 3 + \frac{1}{2}w_3 - \frac{3}{2}x_2$$

$$\boxed{x_2 = 2 + \frac{1}{3}w_3 - \frac{2}{3}w_2}$$

$$x_1 = 5 - \frac{1}{2}w_3 + \frac{1}{2}\left(2 + \frac{1}{3}w_3 - \frac{2}{3}w_2\right) = 6 - \frac{1}{3}w_3 - \frac{1}{3}w_2$$

$$w_1 = 17 - \frac{1}{2}w_3 - \frac{5}{2}\left(2 + \frac{1}{3}w_3 - \frac{2}{3}w_2\right) = 12 - \frac{4}{3}w_3 + \frac{5}{3}w_2$$

III $B\,(w_2=0,\ w_3=0)$

Interchange x_2 and w_2 with

w_2 leaving basic and x_2 entering basic.

max $\zeta = 15 - \frac{3}{2}w_3 + \frac{7}{2}x_2$

subject to
$$x_1 = 5 - \frac{1}{2}w_3 + \frac{x_2}{2}$$
$$w_1 = 17 - \frac{1}{2}w_3 - \frac{5}{2}x_2$$
$$w_2 = 3 + \frac{1}{2}w_3 - \frac{3}{2}x_2$$

$$\boxed{x_2 = 2 + \frac{1}{3}w_3 - \frac{2}{3}w_2}$$

$$\zeta = 15 - \frac{3}{2}w_3 + \frac{7}{2}\left(2 + \frac{1}{3}w_3 - \frac{2}{3}w_2\right)$$
$$= 22 - \frac{1}{3}w_3 - \frac{7}{3}w_2$$

III $\quad B\ (w_2 = 0,\ w_3 = 0)$

$$\max\ \zeta = 22 - \frac{1}{3}w_3 - \frac{7}{3}w_2$$

subject to

$$x_1 = 6 - \frac{1}{3}w_3 - \frac{1}{3}w_2$$

$$w_1 = 12 - \frac{4}{3}w_3 + \frac{5}{3}w_2$$

$$\underbrace{x_2}_{\text{basic}} = \underbrace{2 + \frac{1}{3}w_3 - \frac{2}{3}w_2}_{\text{non-basic}}$$

$$x_1,\ x_2,\ w_1,\ w_2,\ w_3 \geq 0$$

B is optimal!! $\quad \max\ \zeta = 22$

$$\zeta = 22 - \frac{1}{3}w_3 - \frac{7}{3}w_2$$

III $\quad B\,(w_2=0,\ w_3=0)$

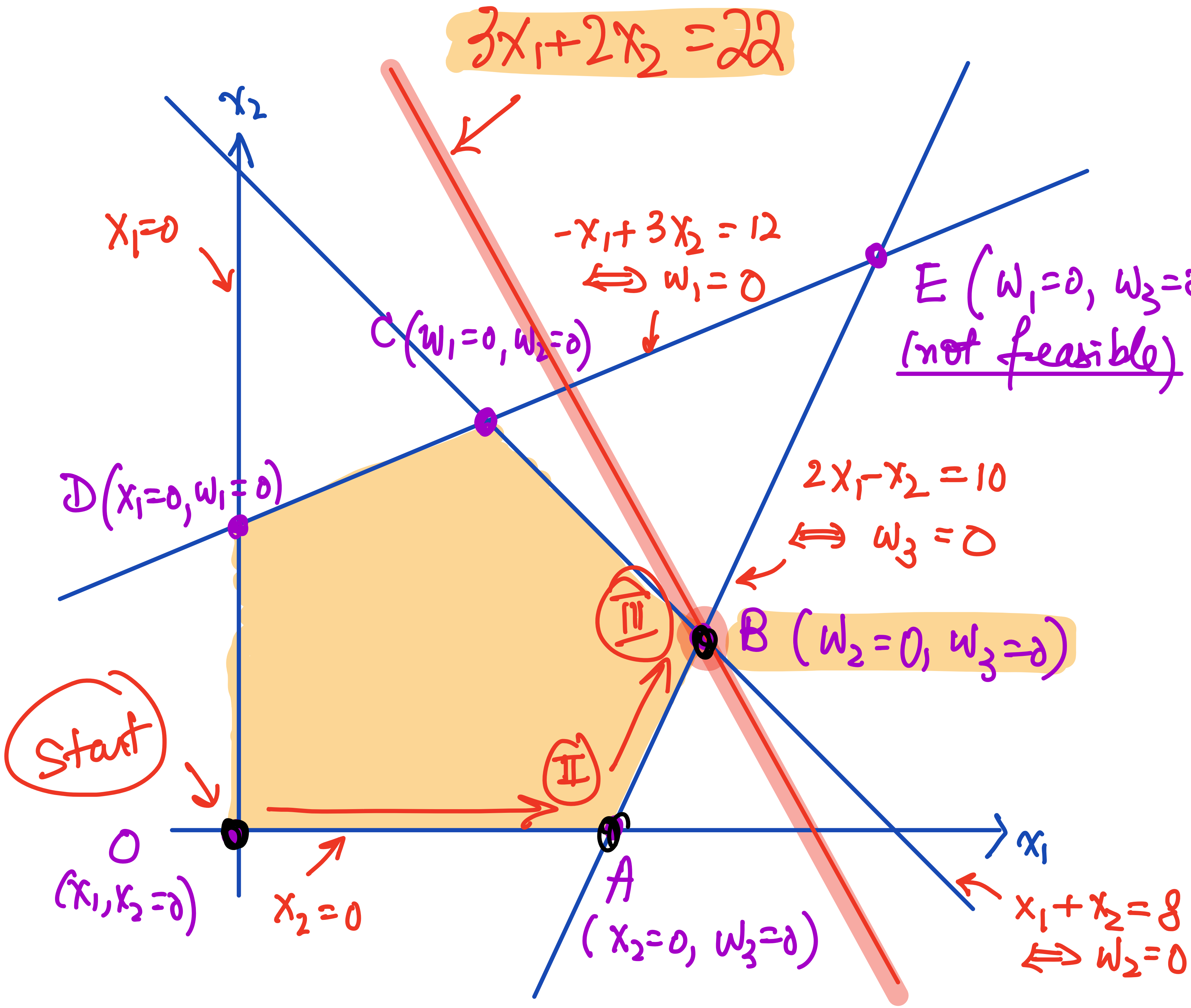
$$\max\ \zeta = 22 - \tfrac{1}{3}w_3 - \tfrac{7}{3}w_2$$

subject to

$$x_1 = 6 - \tfrac{1}{3}w_3 - \tfrac{1}{3}w_2$$
$$w_1 = 12 - \tfrac{4}{3}w_3 + \tfrac{5}{3}w_2$$
$$x_2 = 2 + \tfrac{1}{3}w_3 - \tfrac{2}{3}w_2$$
$$x_1,\ x_2,\ w_1,\ w_2,\ w_3 \geq 0$$

B is optimal!! $\quad \max \zeta = 22$

$$\zeta = 22 - \tfrac{1}{3}w_3 - \tfrac{7}{3}w_2 \leq 22, \quad = 22 \text{ at } w_2=0, w_3=0$$



3x1+2x2 =22
x2
x1=0
-x1+3x2 = 12
⟺ w1 = 0
E (w1=0, w3=0)
(not feasible)
C (w1=0, w2=0)
D (x1=0, w1=0)
2x1-x2 =10
⟺ w3 = 0
III
B (w2=0, w3=0)
Start
II
O
(x1,x2=0)
x2=0
A
(x2=0, w3=0)
x1
x1+x2=8
⟺ w2=0

[V] p. 11

$$\max \quad \zeta = 5x_1 + 4x_2 + 3x_3$$

subject to

$$\begin{cases} 2x_1 + 3x_2 + x_3 \le 5 \\ 4x_1 + x_2 + 2x_3 \le 11 \\ 3x_1 + 4x_2 + 2x_3 \le 8 \\ x_1, x_2, x_3 \ge 0 \end{cases}$$

$$\updownarrow$$

$$\begin{cases} w_1 = 5 - 2x_1 - 3x_2 - x_3 \\ w_2 = 11 - 4x_1 - x_2 - 2x_3 \\ w_3 = 8 - 3x_1 - 4x_2 - 2x_3 \\ x_1, x_2, x_3, w_1, w_2, w_3 \ge 0 \end{cases}$$

[V] p. 11 Set $x_1 = x_2 = x_3 = 0$

I

$$\max \ \zeta = 5x_1 + 4x_2 + 3x_3$$

subject to

$$\begin{cases} 2x_1 + 3x_2 + x_3 \leq 5 \\ 4x_1 + x_2 + 2x_3 \leq 11 \\ 3x_1 + 4x_2 + 2x_3 \leq 8 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

$$\begin{cases} w_1 = 5 - 2x_1 - 3x_2 - x_3 = 5 \geq 0 \\ w_2 = 11 - 4x_1 - x_2 - 2x_3 = 11 \geq 0 \\ w_3 = 8 - 3x_1 - 4x_2 - 2x_3 = 8 \geq 0 \\ x_1, x_2, x_3, w_1, w_2, w_3 \geq 0 \end{cases}$$

[V] p. 11 Set $x_1 = x_2 = x_3 = 0$

$$\max \zeta = 5x_1 + 4x_2 + 3x_3$$

increase x_1, keep $x_2 = x_3 = 0$

subject to

$$w_1 = 5 - 2x_1 - 3x_2 - x_3$$

$$w_2 = 11 - 4x_1 - x_2 - 2x_3$$

$$w_3 = 8 - 3x_1 - 4x_2 - 2x_3$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

$w_3 \geq 0 \Longrightarrow x_1 \leq 8/3$

$w_2 \geq 0 \Longrightarrow x_1 \leq 11/4$

$w_1 \geq 0 \Longrightarrow x_1 \leq 5/2$ ⟵ most strict

[V] p. 11 Set $x_1 = x_2 = x_3 = 0$

$$\max \ \zeta = 5x_1 + 4x_2 + 3x_3$$

increase x_1, keep $x_2 = x_3 = 0$

subject to
$$\begin{cases} w_1 = 5 - 2x_1 - 3x_2 - x_3 \\ w_2 = 11 - 4x_1 - x_2 - 2x_3 \\ w_3 = 8 - 3x_1 - 4x_2 - 2x_3 \\ x_1, x_2, x_3, w_1, w_2, w_3 \geq 0 \end{cases}$$

Set $x_1 = \frac{5}{2}$, $\Rightarrow w_1 = 0$, interchange x_1, w_1

$$\begin{cases} x_1 = \frac{5}{2} - \frac{w_1}{2} - \frac{3}{2}x_2 - \frac{x_3}{2} \\ w_2 = 1 + 2w_1 + 5x_2 \\ w_3 = \frac{1}{2} + \frac{3}{2}w_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 \end{cases}$$

[V] p. 11 Set $x_1 = x_2 = x_3 = 0$

$$\max \ \zeta = 5x_1 + 4x_2 + 3x_3$$

increase x_1, keep $x_2 = x_3 = 0$

subject to

$$\begin{cases} w_1 = 5 - 2x_1 - 3x_2 - x_3 \\ w_2 = 11 - 4x_1 - x_2 - 2x_3 \\ w_3 = 8 - 3x_1 - 4x_2 - 2x_3 \\ x_1, x_2, x_3, w_1, w_2, w_3 \geq 0 \end{cases}$$

Set $x_1 = \frac{5}{2}$, $\Rightarrow w_1 = 0$, interchange x_1, w_1

$$\zeta = 5x_1 + 4x_2 + 3x_3$$

$$\frac{5}{2} - \frac{w_1}{2} - \frac{3}{2}x_2 - \frac{x_3}{2}$$

$$= \frac{25}{2} - \frac{5}{2}w_1 - \frac{7}{2}x_2 + \frac{1}{2}x_3$$

[V] p. 11 Set $w_1=0,\ x_2=0,\ x_3=0$

II

$$\max.\ \zeta = \frac{25}{2} - \frac{5}{2}w_1 - \frac{7}{2}x_2 + \frac{1}{2}x_3$$

subj.

$$\underbrace{\begin{aligned} x_1 \\ w_2 \\ w_3 \end{aligned}}_{\text{basic}} \begin{aligned} &= \underbrace{\frac{5}{2} - \frac{w_1}{2} - \frac{3}{2}x_2 - \frac{x_3}{2} \\ &= 1 + 2w_1 + 5x_2 \\ &= \frac{1}{2} + \frac{3}{2}w_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3}_{\text{non-basic}} \end{aligned}$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

[V] p. 11 Set $w_1 = 0,\ x_2 = 0,\ x_3 = 0$

$$\max.\ \zeta = \frac{25}{2} - \frac{5}{2}w_1 - \frac{7}{2}x_2 + \frac{1}{2}x_3$$

increase x_3, keep $w_1, x_2 = 0$

subj.

$$x_1 = \frac{5}{2} - \frac{w_1}{2} - \frac{3}{2}x_2 - \frac{x_3}{2}$$

$$w_2 = 1 + 2w_1 + 5x_2$$

$$\underbrace{w_3}_{\text{basic}} = \underbrace{\frac{1}{2} + \frac{3}{2}w_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3}_{\text{non-basic}}$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

$w_3 \geq 0 \Longrightarrow x_3 \leq 1 \longleftarrow$ most strict.

$w_2 \geq 0 \Longrightarrow x_3$ can be any number

$x_1 \geq 0 \Longrightarrow x_3 \leq 5$

[V] p. 11 Set $w_1=0,\ x_2=0,\ w_3=0$

III

$$\max.\ \zeta = \frac{25}{2} - \frac{5}{2}w_1 - \frac{7}{2}x_2 + \frac{1}{2}x_3$$

increase x_3, keep $w_1, x_2 = 0$

subj.

$$x_1 = \frac{5}{2} - \frac{w_1}{2} - \frac{3}{2}x_2 - \frac{x_3}{2}$$

$$w_2 = 1 + 2w_1 + 5x_2$$

$$\underbrace{w_3}_{\text{basic}} = \underbrace{\frac{1}{2} + \frac{3}{2}w_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3}_{\text{non-basic}}$$

Set $x_3 = 1 \Rightarrow w_3 = 0$. Interchange x_3, w_3

$$x_3 = 1 + 3w_1 + x_2 - 2w_3$$

$$x_1 = 2 - 2w_1 - 2x_2 + w_3$$

$$w_2 = 1 + 2w_1 + 5x_2$$

[V] p. 11 Set $w_1=0,\ x_2=0,\ w_3=0$

increase x_3, keep $w_1, x_2 = 0$

max. $\zeta = \frac{25}{2} - \frac{5}{2}w_1 - \frac{7}{2}x_2 + \frac{1}{2}x_3$

subj.

$$x_1 = \frac{5}{2} - \frac{w_1}{2} - \frac{3}{2}x_2 - \frac{x_3}{2}$$

$$w_2 = 1 + 2w_1 + 5x_2$$

$$\underbrace{w_3}_{\text{basic}} = \underbrace{\frac{1}{2} + \frac{3}{2}w_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3}_{\text{non-basic}}$$

Set $x_3 = 1 \Rightarrow w_3 = 0$. Interchange x_3, w_3

$x_3 = 1 + 3w_1 + x_2 - 2w_3$

$$\zeta = \frac{25}{2} - \frac{5}{2}w_1 - \frac{7}{2}x_2 + \frac{1}{2}x_3$$

$$= 13 - w_1 - 3x_2 - w_3$$

[V] p. 11 Set $w_1=0,\ x_2=0,\ w_3=0$

$$\max\ \zeta = \frac{25}{2} - \frac{5}{2}w_1 - \frac{7}{2}x_2 + \frac{1}{2}x_3$$

increase x_3, keep $w_1, x_2 = 0$

sub.
$$x_1 = \frac{5}{2} - \frac{w_1}{2} - \frac{3}{2}x_2 - \frac{x_3}{2}$$
$$w_2 = 1 + 2w_1 + 5x_2$$
$$\underbrace{w_3}_{\text{basic}} = \underbrace{\frac{1}{2} + \frac{3}{2}w_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3}_{\text{non-basic}}$$

Set $x_3 = 1 \Rightarrow w_3 = 0$. Interchange x_3, w_3

$$\zeta = 13 - w_1 - 3x_2 - w_3 \leq 13$$

max $\zeta = 13$, achieved at $w_1 = x_2 = w_3 = 0$

Concept of a Dictionary of Variables

Initially $(\underbrace{x_1, x_2, \ldots, x_n}_{n \text{ non-basic vars}}, \underbrace{w_1, w_2, \ldots, w_m}_{m \text{ basic variables}})$

$$(\underbrace{x_1, x_2, \ldots, x_n}, \underbrace{x_{n+1}, x_{n+2}, \ldots, x_{n+m}})$$

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \leq \begin{pmatrix} x_{n+1} \\ x_{n+2} \\ \vdots \\ x_{n+m} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} - \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{ij} & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$\zeta = c_0 + c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

Concept of a Dictionary of Variables

Initially $(\underbrace{x_1, x_2, \ldots, x_n}_{n \text{ non-basic vars}}, \underbrace{w_1, w_2, \ldots, w_m}_{m \text{ basic variables}})$

$$\underbrace{(\underbrace{x_1, x_2, \ldots, x_n}, \underbrace{x_{n+1}, x_{n+2}, \ldots x_{n+m}})}$$

interchange one basic and one non-basic variable.

Concept of a Dictionary of Variables

$$(\underbrace{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n}_{n \text{ non-basic}}, \underbrace{\bar{x}_{n+1}, \bar{x}_{n+2}, \ldots \bar{x}_{n+m}}_{m \text{ basic}})$$

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \leq \begin{pmatrix} \bar{x}_{n+1} \\ \bar{x}_{n+2} \\ \vdots \\ \bar{x}_{n+m} \end{pmatrix} = \begin{pmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \vdots \\ \bar{b}_m \end{pmatrix} - \begin{pmatrix} \bar{a}_{11} & \bar{a}_{12} & \cdots & \bar{a}_{1n} \\ & & \bar{a}_{ij} & \\ \bar{a}_{m1} & \bar{a}_{m2} & \cdots & \bar{a}_{mn} \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_n \end{pmatrix}$$

$$\zeta = \bar{c}_0 + \bar{c}_1 \bar{x}_1 + \bar{c}_2 \bar{x}_2 + \cdots + \bar{c}_n \bar{x}_n$$

Concept of a Dictionary of Variables

(1) In each dictionary, setting n non-basic variables to zero, corresponds to a vertex in a polyhedron.

(2) Total number of dictionaries

$$= \binom{n+m}{n} = \frac{(n+m)!}{n!\,m!}$$

= "# of vertices"

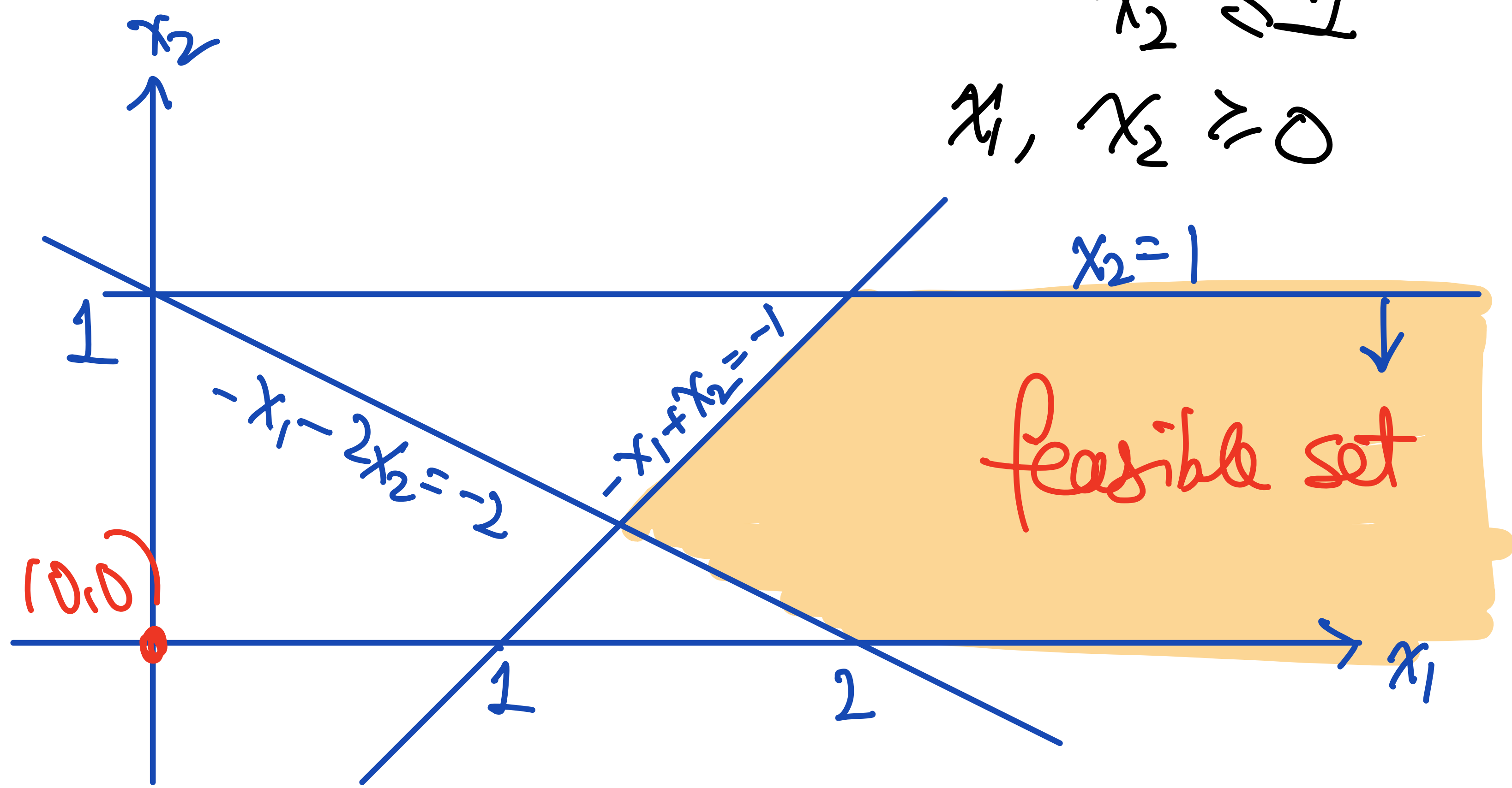
(Note: not all vertices are feasible)

What if the origin is not feasible?

[V] p.18

$$\max\ \zeta = -2x_1 - x_2$$

subject to
$$-x_1 + x_2 \le -1$$
$$-x_1 - 2x_2 \le -2$$
$$x_2 \le 1$$
$$x_1, x_2 \ge 0$$



Auxiliary Problem

$$\max \ \xi = -x_0$$

subject to

$$-x_1 + x_2 - x_0 \leq -1$$
$$-x_1 - 2x_2 - x_0 \leq -2$$
$$x_2 - x_0 \leq 1$$
$$x_1, x_2, x_0 \geq 0$$

(1) Must be feasible : choose x_0 large enough

(2) Original problem is feasible

$\iff \max \xi = 0$ (Note: $\max \xi \leq 0$)

Auxiliary Problem

$$\max \ \xi = -x_0$$

subject to

$$w_1 = -1 + x_1 - x_2 + x_0$$
$$w_2 = -2 + x_1 + 2x_2 + x_0$$
$$w_3 = 1 - x_2 + x_0$$

Setting $x_1, x_2, x_0 = 0$

$w_1 = -1$

$w_2 = -2$ ← most infeasible

$w_3 = 1$ (interchange x_0 & w_2)

Auxiliary Problem

$$\max \ \xi = -x_0$$

subject to

$$w_1 = -1 + x_1 - x_2 + x_0$$

$$w_2 = -2 + x_1 + 2x_2 + x_0$$

$$w_3 = 1 - x_2 + x_0$$

$$\boxed{x_0 = 2 - x_1 - 2x_2 + w_2}$$

$$w_1 = -1 + x_1 - x_2 + 2 - x_1 - 2x_2 + w_2$$

$$= 1 - 3x_2 + w_2$$

$$w_3 = 1 - x_2 + 2 - x_1 - 2x_2 + w_2$$

$$= 3 - x_1 - 3x_2 + w_2$$

Auxiliary Problem

I

$$\max \ \xi = -2 + x_1 + 2x_2 - w_2$$

subject to

$$x_0 = 2 - x_1 - 2x_2 + w_2$$

$$w_1 = 1 - 3x_2 + w_2$$

$$w_3 = \underbrace{3 - x_1 - 3x_2 + w_2}_{\text{Set } x_1, x_2, w_2 = 0}$$

feasible !!

Auxiliary Problem

$$\max \ \xi = -2 + x_1 + 2x_2 - w_2$$

subject to

$$x_0 = 2 - x_1 - 2x_2 + w_2$$

$$w_1 = 1 - 3x_2 + w_2$$

$$w_3 = 3 - \underbrace{x_1 - 3x_2 + w_2}_{}$$

Set $x_1, x_2, w_2 = 0$

feasible !!

increase x_2

x_0 : $x_2 \leq 1$

w_1 : $x_2 \leq 1/3$ ← interchange x_2, w_1

w_3 : $x_2 \leq 1$

Auxiliary Problem

$$\max \ \xi = -2 + x_1 + 2x_2 - w_2$$

subject to

$$x_0 = 2 - x_1 - 2x_2 + w_2$$

$$w_1 = 1 - 3x_2 + w_2$$

$$w_3 = 3 - x_1 - 3x_2 + w_2$$

$$\boxed{x_2 = \frac{1}{3} - \frac{1}{3}w_1 + \frac{1}{3}w_2}$$

$$w_3 = 2 - x_1 + w_1$$

$$x_0 = \frac{4}{3} - x_1 + \frac{2}{3}w_1 + \frac{1}{3}w_2$$

Auxiliary Problem

$$\max \ \xi = -2 + x_1 + 2x_2 - w_2$$

subject to

$$x_0 = 2 - x_1 - 2x_2 + w_2$$

$$w_1 = 1 - 3x_2 + w_2$$

$$w_3 = 3 - x_1 - 3x_2 + w_2$$

$$\boxed{x_2 = \frac{1}{3} - \frac{1}{3}w_1 + \frac{1}{3}w_2}$$

$$\xi = -\frac{4}{3} + x_1 - \frac{2}{3}w_1 - \frac{w_2}{3}$$

Auxiliary Problem

II

max $\zeta = -\frac{4}{3} + x_1 - \frac{2}{3} w_1 - \frac{w_2}{3}$

subject to

$$x_2 = \frac{1}{3} - \frac{1}{3} w_1 + \frac{1}{3} w_2$$

$$w_3 = 2 - x_1 + w_1$$

$$x_0 = \frac{4}{3} - x_1 + \frac{2}{3} w_1 + \frac{1}{3} w_2$$

increase x_1

w_3: $x_1 \le 2$

x_0: $x_1 \le \frac{4}{3}$ $\leftarrow$ interchange

x_0, x_1

Auxiliary Problem

II

max $\zeta = -\frac{4}{3} + x_1 - \frac{2}{3} w_1 - \frac{w_2}{3}$

subject to

$$x_2 = \frac{1}{3} - \frac{1}{3} w_1 + \frac{1}{3} w_2$$

$$w_3 = 2 - x_1 + w_1$$

$$x_0 = \frac{4}{3} - x_1 + \frac{2}{3} w_1 + \frac{1}{3} w_2$$

$$\boxed{x_1 = \frac{4}{3} + \frac{2}{3} w_1 + \frac{1}{3} w_2 - x_0}$$

$$w_3 = \frac{2}{3} + \frac{1}{3} w_1 - \frac{1}{3} w_2 + x_0$$

$$x_2 = \frac{1}{3} - \frac{1}{3} w_1 + \frac{1}{3} w_2$$

Auxiliary Problem

II

$$\max \quad \xi = -\frac{4}{3} + x_1 - \frac{2}{3} w_1 - \frac{w_2}{3}$$

subject to

$$x_2 = \frac{1}{3} - \frac{1}{3} w_1 + \frac{1}{3} w_2$$

$$w_3 = 2 - x_1 + w_1$$

$$x_0 = \frac{4}{3} - x_1 + \frac{2}{3} w_1 + \frac{1}{3} w_2$$

$$\boxed{x_1 = \frac{4}{3} + \frac{2}{3} w_1 + \frac{1}{3} w_2 - x_0}$$

$$w_3 = \frac{2}{3} + \frac{1}{3} w_1 - \frac{1}{3} w_2 + x_0$$

$$x_2 = \frac{1}{3} - \frac{1}{3} w_1 + \frac{1}{3} w_2$$

$$\xi = -\frac{4}{3} + x_1 - \frac{2}{3} w_1 - \frac{w_2}{3}$$

$$= -x_0$$

Auxiliary Problem

III

$$\max \quad \xi = -x_0$$

subject to

$$x_1 = \frac{4}{3} + \frac{2}{3}w_1 + \frac{1}{3}w_2 - x_0$$

$$w_3 = \frac{2}{3} + \frac{1}{3}w_1 - \frac{1}{3}w_2 + x_0$$

$$x_2 = \frac{1}{3} - \frac{1}{3}w_1 + \frac{1}{3}w_2$$

Setting $w_1 = w_2 = x_0 = 0$

$$\max \quad \xi = -x_0 = 0$$

Optimal!

Back to Original Problem

$$\max \zeta = -2x_1 - x_2$$

subject to

$$\begin{cases} -x_1 + x_2 \le -1 \\ -x_1 - 2x_2 \le -2 \\ x_2 \le 1 \\ x_1, x_2 \ge 0 \end{cases}$$

We already have

$$\begin{cases} x_1 = \frac{4}{3} + \frac{2}{3}w_1 + \frac{1}{3}w_2 - \cancel{x_0} \quad 0 \\ w_3 = \frac{2}{3} + \frac{1}{3}w_1 - \frac{1}{3}w_2 + \cancel{x_0} \quad 0 \\ x_2 = \frac{1}{3} - \frac{1}{3}w_1 + \frac{1}{3}w_2 \end{cases}$$

Back to Original Problem

$$\max \zeta = -2x_1 - x_2$$

$$\begin{cases} x_1 = \frac{4}{3} + \frac{2}{3}w_1 + \frac{1}{3}w_2 \\ w_3 = \frac{2}{3} + \frac{1}{3}w_1 - \frac{1}{3}w_2 \\ x_2 = \frac{1}{3} - \frac{1}{3}w_1 + \frac{1}{3}w_2 \end{cases}$$

$$x_1, x_2, w_1, w_2, w_3 \geq 0$$

$$\zeta = -2x_1 - x_2$$

$$= -3 - w_1 - w_2$$

Already at Optimal

$(w_1 = w_2 = 0)$