Primal and Dual Problems

(P)

$$\boxed{\begin{array}{ll} \max & \xi(x) = c^T x \\ \text{subject to} & Ax \leq b \\ & x \geq 0 \end{array}}$$

(D)

$$\boxed{\begin{array}{ll} \min & \xi(y) = b^T y \\ \text{subject to} & A^T y \geq c \\ & y \geq 0 \end{array}} \iff \boxed{\begin{array}{ll} \max & -\xi(y) = -b^T y \\ \text{subject to} & (-A^T) y \leq -c \\ & y \geq 0 \end{array}}$$

Primal and Dual Problems

(P) $\boxed{\begin{array}{ll} \max & \zeta(x) = c^T X \\ \text{subject to} & AX \le b \\ & X \ge 0 \end{array}}$ $\sim \sum_j c_j x_j$

$$\sum_i y_i \left(\sum_j a_{ij} x_j \le b_i \right), \quad y_i \ge 0$$

$$\sum_j \underbrace{\left(\sum_i a_{ij} y_i \right)}_{c_j \le} x_j \le \sum_i b_i y_i$$

Thm 5.1 Weak Duality

$$\boxed{\zeta(X) = \sum_j c_j x_j \le \sum_i b_i y_i = \xi(Y)}$$

Matrix Representations

$$(P) \quad \max \; \zeta(x) = \zeta_0 + c^T x$$

$$w = b - Ax \geq 0$$

$$x \geq 0$$

$$\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{pmatrix}$$

$$\begin{array}{c|cccc}
\zeta_0 & c_1 & c_2 & \cdots & c_n \\
\hline
b_1 & -a_{11} & -a_{12} & \cdots & -a_{1n} \\
b_2 & -a_{21} & -a_{22} & \cdots & -a_{2n} \\
\vdots & & \vdots & & \vdots \\
b_m & -a_{m1} & -a_{m2} & \cdots & -a_{mn}
\end{array}$$

$$(D) \quad \max \; -\xi(y) = -\xi_0 - b^T y$$

$$z = -c + A^T y \geq 0$$

$$y \geq 0$$

$$\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}$$

$$\begin{array}{c|cccc}
-\xi_0 & -b_1 & -b_2 & \cdots & -b_m \\
\hline
-c_1 & a_{11} & a_{21} & \cdots & a_{m1} \\
-c_2 & a_{12} & a_{22} & \cdots & a_{m2} \\
\vdots & & & & \\
-c_n & a_{1n} & a_{2n} & \cdots & a_{mn}
\end{array}$$

Matrix Representations

(P) $\max \ \zeta(x) = \zeta_0 + c^T x$

$$\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{pmatrix} \qquad w = b - Ax \geq 0$$

$$x \geq 0$$

$$\begin{array}{c|c} \zeta_0 & c^T \\ \hline b & -A \end{array}$$

negative transpose

(D) $\max \ -\xi(y) = -\xi_0 - b^T y$

$$\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} \qquad z = -c + A^T y \geq 0$$

$$y \geq 0$$

$$\begin{array}{c|c} -\xi_0 & -b^T \\ \hline -c & A^T \end{array}$$

Complementary Variables

$$(x_1, x_2, \ldots, x_n, w_1, w_2, \ldots w_m)$$

$y_i \sim w_i$ $\qquad$ $x_j \sim z_j$

$$(y_1, y_2, \ldots y_m, z_1, z_2, \ldots z_n)$$

(P) $\qquad$ x_j enter (leaves), w_i enter (leaves)

$$\Updownarrow$$

(D) $\qquad$ z_j leaves (enters), y_i leaves (enters)

Neg. Transpose Property is Preserved in each Simplex Step

[V] p.64

(P)

ζ	$=$		$+$	4	x_1	$+$	1	x_2	$+$	3	x_3
w_1	$=$	1	$-$		x_1	$-$	4	x_2			
w_2	$=$	3	$-$	3	x_1	$+$		x_2	$-$		x_3

x_3 enters, w_2 leaves (greatest increase rule)

neg. transpose

(D)

$-\xi$	$=$		$-$	1	y_1	$-$	3	y_2
z_1	$=$	-4	$+$		y_1	$+$	3	y_2
z_2	$=$	-1	$+$	4	y_1	$-$		y_2
z_3	$=$	-3				$+$		y_2

z_3 leaves, y_2 enters

Neg. Transpose Property is Preserved in each Simplex Step

[V] p.64

$(P)_2$

$$
\begin{array}{rcrcrlcrlcrl}
\zeta & = & 9 & - & 5 & x_1 & + & 4 & x_2 & - & 3 & w_2 \\
\hline
w_1 & = & 1 & - & & x_1 & - & 4 & x_2 & + & & \\
x_3 & = & 3 & - & 3 & x_1 & + & & x_2 & - & & w_2
\end{array}
$$

x_2 enters, w_1 leaves

neg. transpose

$(D)_2$

$$
\begin{array}{rcrcrlcrl}
-\xi & = & -9 & - & 1 & y_1 & - & 3 & z_3 \\
\hline
z_1 & = & 5 & + & & y_1 & + & 3 & z_3 \\
z_2 & = & -4 & + & 4 & y_1 & - & & z_3 \\
y_2 & = & 3 & + & & & + & & z_3
\end{array}
$$

z_2 leaves, y_1 enters

Neg. Transpose Property is Preserved in each Simplex Step

[V] p.64

(P)

$$\begin{array}{rcrcrcrcr} \zeta & = & 10 & - & 6 & x_1 & - 1 & w_1 & - 3 \; w_2 \\ \hline x_2 & = & 0.25 & - & 0.25 & x_1 & - 0.25 & w_1 & \\ x_3 & = & 3.25 & - & 3.25 & x_1 & - 0.25 & w_1 & - \; w_2 \end{array}$$

Optimal for (P)

neg. transpose

(D)

$$\begin{array}{rcrcrcrcr} -\xi & = & -10 & - & 0.25 & z_2 & - 3.25 & z_3 \\ \hline z_1 & = & 6 & + & 0.25 & z_2 & + 3.25 & z_3 \\ y_1 & = & 1 & + & 0.25 & z_2 & + 0.25 & z_3 \\ y_2 & = & 3 & & & & + 1 & z_3 \end{array}$$

Optimal for (D)

Optimal values for y_1, y_2 $\left(\begin{array}{l} z_2, \; z_3 = 0, \\ y_1 = 1, \; y_2 = 3, \; z_1 = 0 \end{array} \right)$

Neg. Transpose Property is Preserved in each Simplex Step

at Optimal:

$c_j^* \le 0$

(P)

$$
\begin{array}{c|cccc}
\zeta^* & c_1^* & c_2^* & \cdots & c_n^* \\
\hline
b_1^* & -a_{11}^* & -a_{12}^* & \cdots & -a_{1n}^* \\
b_2^* & & & & \\
\vdots & & -a_{ij}^* & & \\
b_m^* & -a_{m1}^* & -a_{m2}^* & \cdots & -a_{mn}^*
\end{array}
$$

$b_i^* \ge 0$

Neg. Transpose Property is Preserved in each Simplex Step

at Optimal:

$-b_i^* \le 0$

$$
(D)\quad
\begin{array}{c|cccc}
-\zeta^* & -b_1^* & -b_2^* & \cdots & -b_m^* \\
\hline
-c_1^* & a_{11}^* & a_{21}^* & \cdots & a_{m1}^* \\
-c_2^* & & & & \\
\vdots & & a_{ji}^* & & \\
-c_n^* & a_{1n}^* & a_{2n}^* & \cdots & a_{mn}^*
\end{array}
$$

Strong Duality

$-c_j^* \ge 0$

Strong Duality Theorem

If (P) has an optimal solution (x_j^*) then (D) also has an optimal solution (y_i^*) and

$$\left(\zeta^* = \right) \sum_j c_j x_j^* = \sum_i b_i y_i^* \left(= \xi^*\right)$$

i.e. duality gap $= 0$

(The proof gives a formula for y_i^*)

Strong Duality Theorem

Pf

$$\left(\begin{array}{l} N : \text{non-basic vars} \\ B : \text{basic vars} \end{array} \right)$$

① $\zeta = \sum_j \bar{c}_j \bar{x}_j + \sum_i 0\, \bar{w}_i$ (original formulation)

$\qquad\qquad\qquad N \qquad\qquad B$

Strong Duality Theorem

Pf

② at (P)'s optimal

$$\zeta = \zeta^* + \sum_j \bar{c}_j \bar{x}_j + \sum_i 0 \, \bar{w}_i$$

$(x_1, x_2, \ldots, x_n, w_1, \ldots w_m)$

$$= \zeta^* + \sum_j c_j^* x_j + \sum_i d_i^* w_i$$

$(c_j^* \le 0)$ $(d_i^* \le 0)$

$$= \zeta^* + \sum_j c_j^* x_j + \sum_i d_i^* \left(b_i - \sum_j a_{ij} x_j \right)$$

Strong Duality Theorem

Pf ② at (P)'s optimal

$$\zeta = \zeta^* + \sum_j c_j^* x_j + \sum_i d_i^* \left(b_i - \sum_j a_{ij} x_j\right)$$

$$= \left(\zeta^* + \sum_i d_i^* b_i\right)$$

$$+ \sum_j \left(c_j^* - \sum_i a_{ij} d_i^*\right) x_j$$

Strong Duality Theorem

Pf Compare ① and ②

Const term: $0 = \zeta^* + \sum_i d_i^* b_i$

x_j : $c_j = c_j^* - \sum_i a_{ij} d_i^*$

Let $y_i^* = -d_i^*$ (≥ 0). Then

$$\boxed{\zeta^* = \sum_i b_i y_i^*}$$

Strong Duality Theorem

Pf Compare ① and ②

Const term: $0 = \zeta^* + \sum_i d_i^* b_i$

x_j : $c_j = c_j^* - \sum_i a_{ij} d_i^*$

$$c_j = c_j^* + \sum_i a_{ij} y_i^* \leq \sum_i a_{ij} y_i^*$$

$(c_j^* \leq 0)$

Strong Duality Theorem

Pf

Hence there exists an y^* s.t.

$$\boxed{\zeta^* = \zeta(x^*) = \xi(y^*)}$$

This must be opt. for (D)

(By Weak Duality,

$\zeta(x^*) \leq \xi(y^*)$)

Four and Only Four Possibilities

(1) (P) has an optimal solution

$\Rightarrow$ (D) also has an optimal solution

$\zeta^* = \xi^*$, duality gap $= 0$

(2) (P) is feasible but unbounded

$\Rightarrow$ (D) must be infeasible

$\zeta(x) \leq \xi(y)$

$\rightarrow +\infty$

$\zeta^* = \xi^* = +\infty$

(min empty set $= +\infty$)

Four and Only Four Possibilities

(3) (D) is feasible but unbounded

$\Longrightarrow$ (P) must be infeasible

$$\zeta(x) \le \xi(y)$$

$\to -\infty$

$\zeta^* = \xi^* = -\infty$

(max empty set $= -\infty$)

(4) Both (P) and (D) are infeasible

$$\Longrightarrow \quad \zeta^* = -\infty, \quad \xi^* = +\infty$$

(duality gap $= +\infty$)

Four and Only Four Possibilities

[V, p.69]

(P)

$$\begin{aligned} \text{maximize} \quad & 2x_1 - x_2 \\ \text{subject to} \quad & x_1 - x_2 \le 1 \\ & -x_1 + x_2 \le -2 \\ & x_1,\ x_2 \ge 0. \end{aligned}$$

inconsistent

(D)

$$\begin{aligned} \min \quad & y_1 - 2y_2 \\ \text{subject to} \quad & y_1 - y_2 \geq 2 \\ & -y_1 + y_2 \geq -1 \\ & y_1,\ y_2 \geq 0 \end{aligned}$$

inconsistent

Complementary Slackness

[P]

THEOREM 5.3. *Suppose that* $x = (x_1, x_2, \ldots, x_n)$ *is primal feasible and that* $y = (y_1, y_2, \ldots, y_m)$ *is dual feasible. Let* $(w_1, w_2, \ldots, w_m)$ *denote the corresponding primal slack variables, and let* $(z_1, z_2, \ldots, z_n)$ *denote the corresponding dual slack variables. Then* x *and* y *are optimal for their respective problems if and only if*

$$
\begin{aligned}
x_j z_j &= 0, \qquad \text{for } j = 1, 2, \ldots, n, \\
w_i y_i &= 0, \qquad \text{for } i = 1, 2, \ldots, m.
\end{aligned} \tag{5.7}
$$

(pairing of variables)

i.e. either x_j or $z_j = 0$ $\quad j = 1, \ldots, n$

and

either y_i or $w_i = 0$ $\quad i = 1, \ldots, m$

Complementary Slackness

Pf $$\sum_j c_j x_j \le \sum_j \Big(\sum_i a_{ij} y_i\Big) x_j$$

$$= \sum_i \Big(\sum_j a_{ij} x_j\Big) y_i \le \sum_i b_i y_i$$

$$\sum_j c_j x_j \overset{1}{\underset{=}{\le}} \sum_j \Big(\sum_i a_{ij} y_i\Big) x_j$$

$$= \sum_i \Big(\sum_j a_{ij} x_j\Big) y_i \overset{2}{\underset{=}{\le}} \sum_i b_i y_i$$

$=$

Complementary Slackness

pf

$$\sum_j c_j x_j \overset{①}{=} \sum_j \Big(\sum_i a_{ij} y_i \Big) x_j$$

$$0 = \sum_j \underbrace{\Big(\sum_i a_{ij} y_i - c_j \Big)}_{z_j} x_j$$

$$= \sum_j z_j x_j \geq 0$$

$$\Longrightarrow \quad z_j x_j = 0$$

Complementary Slackness

pf $$\sum_i \left(\sum_j a_{ij} x_j\right) y_i \overset{②}{=} \sum_i b_i y_i$$

$$0 = \sum_i \left(\underbrace{b_i - \sum_j a_{ij} x_j}_{w_i}\right) y_i$$

$$= \sum_i w_i y_i \geq 0$$

$$\Longrightarrow \quad w_i y_i = 0$$