Duality (General Form)

$$\underset{x \in R^d}{\text{maximize}} \quad \zeta = f_0(x)$$

6

$$\text{subject to} \quad f_i(x) \leq 0 \qquad i = 1, \dots, m$$

$$g_j(x) = 0 \qquad j = 1, \dots, n$$

$$\underset{x \in R^d}{\text{maximize}} \quad \zeta = f_0(x)$$

6

$$\text{subject to} \quad g_j(x) = 0 \qquad j = 1, \dots, n$$

Equality Constraints

$$\begin{aligned} &\underset{x \in \mathbb{R}^d}{\text{maximize}} \quad \zeta = f_0(x) \\ &\text{subject to} \quad g_j(x) = 0 \qquad j = 1, \ldots, n \end{aligned}$$

6

Lagrange Multiplier (μ) & Lagrangian (L)

$$\underset{x, \mu_1, \ldots, \mu_n}{\text{maximize}} \quad L(x, \mu) = f_0(x) - \sum_{j=1}^{n} \mu_j g_j(x)$$

no constraints

Equality Constraints

G

Compare max f_0 with constraints

& max L with no constraints

Let $\tilde{x} \in G$, i.e. $g_j(\tilde{x}) = 0$

$$f_0(\tilde{x}) = f_0(\tilde{x}) - \sum_j \mu_j \underbrace{g_j(\tilde{x})}_{=0}$$

$$\leq \max_x \left[f_0(x) - \sum_j \mu_j g_j(x) \right]$$

$$= \max_x L(x, \mu) := \xi(\mu)$$

Equality Constraints

Compare $\max f_0$ with constraints

& $\max L$ with no constraints

Let $\tilde{x} \in G$, i.e. $g_j(\tilde{x}) = 0$

$$\zeta(\tilde{x}) = f_0(\tilde{x}) \leq \xi(\mu)$$

any $\tilde{x}$ satisfies constraint

no constraint for μ

$$\zeta^* = \boxed{\max_{\tilde{x} \in G} \zeta(\tilde{x}) \leq \min_{\mu} \xi(\mu)} = \xi^*$$

Equality Constraints

Compare max f_0 with constraints

& max L with no constraints

Let $\tilde{x} \in G$, i.e. $g_j(\tilde{x}) = 0$

$$\zeta(\tilde{x}) = f_0(\tilde{x}) \leq \xi(\mu)$$

any $\tilde{x}$ satisfies constraint

no constraint for μ

$$\boxed{\zeta^* \leq \xi^* \text{ (Weak Duality)}}$$

Equality Constraints

(P) $$\max_{x} \ \zeta(x) = f_0(x)$$

$$\text{s.t.} \quad g_j(x) = 0 \qquad j = 1, 2, \cdots n$$

G

(D) $$\min_{\mu} \ \xi(\mu)$$

no constraint

$L(x, \mu)$

where $$\xi(\mu) = \max_{x} \left[f_0(x) - \sum_j \mu_j g_j(x) \right]$$

no constraint.

Weak Duality $$\boxed{\zeta^* = \max_{x \in G} \zeta(x) \le \min_{\mu} \xi(\mu) = \xi^*}$$

Inequality Constraints

$$\boxed{\begin{array}{lll} \underset{x \in \mathbb{R}^d}{\text{maximize}} & \zeta = f_0(x) & \\ \text{subject to} & f_i(x) \le 0 & i = 1, \cdots, m \\ & g_j(x) = 0 & j = 1, \cdots, n \end{array}}$$

6

$\lambda_i \geq 0$

$$L(x, \lambda, \mu) = f_0(x) - \sum_{i=1}^{m} \lambda_i f_i(x) - \sum_{j=1}^{n} \mu_j g_j(x)$$

Lagrangian fct

Inequality Constraints

Let $\tilde{x} \in \mathcal{C}$, i.e. constraints are satisfied.

$\lambda \geq 0$

$$f_0(\tilde{x}) \leq f_0(\tilde{x}) - \sum_i \underbrace{\lambda_i}_{\geq 0} \underbrace{f_i(\tilde{x})}_{\leq 0} - \sum_j \mu_j \underbrace{g_j(\tilde{x})}_{= 0}$$

$$= L(\tilde{x}, \lambda, \mu)$$

$$\leq \max_{x} L(x, \lambda, \mu) = \xi(\lambda, \mu)$$

(no constraint)

$(\lambda \geq 0)$

Inequality Constraints

Let $\tilde{x} \in \mathcal{G}$, i.e. constraints are satisfied.

$\lambda \geqslant 0$

$$f_0(\tilde{x}) \leqslant \xi(\lambda, \mu)$$

$\tilde{x} \in \mathcal{G}$ $\qquad$ $\lambda \geqslant 0$

$$\underline{\max_{x \in \mathcal{G}}} f_0(x) \leqslant \underline{\min_{\lambda \geqslant 0, \mu}} \xi(\lambda, \mu)$$

$$\boxed{\zeta^* \leqslant \xi^*}$$

Weak Duality

Inequality Constraints

(P) $\quad \max \; \zeta(x) = f_0(x)$ $\qquad G$

$\quad$ s.t. $\quad f_i(x) \leq 0 \qquad i = 1, 2, \cdots, m$

$\qquad\qquad g_j(x) = 0 \qquad j = 1, 2, \cdots, n$

(D) $\quad \min\limits_{\lambda \geq 0, \mu} \; \xi(\lambda, \mu)$

constraint for λ $\qquad L(x, \lambda, \mu)$

$$\xi(\lambda, \mu) = \max_{x} \left[f_0(x) - \sum_i \lambda_i f_0(x) - \sum_j \mu_j g_j(x) \right]$$

no constraint

$$\boxed{\text{Weak Duality:} \quad \zeta^* = \max_{x \in G} \zeta(x) \leq \min_{\lambda \geq 0, \mu} \xi(\lambda, \mu) = \xi^*}$$

Duality for General LP

$$\max f(x) = m^T x$$

s.t.

$$\left.\begin{aligned} A x &\le \vec{b} \\ B x &= \vec{c} \end{aligned}\right\} 6$$

$$L(x, \lambda, \mu) = m^T x - \underbrace{\lambda^T}_{\lambda \geq 0,\ \mathcal{D}}(Ax - b) - \mu^T (Bx - c)$$

$$g(\lambda, \mu) = \max_{x} L(x, \lambda, \mu)$$

no constraint

Duality for General LP

$$\max f(x) = m^T x$$

$$\text{s.t.} \quad \left.\begin{aligned} Ax &\leq \vec{b} \\ Bx &= \vec{c} \end{aligned}\right\} G$$

$$L(x, \lambda, \mu) = m^T x - \underbrace{\lambda^T}_{\lambda \geq 0,\ \varnothing}(Ax - b) - \mu^T(Bx - c)$$

$$x \in G \qquad \boxed{f(x) \leq \xi(\lambda, \mu)} \qquad \lambda \geq 0$$

Duality for General LP

$$\max \zeta(x) = m^T X$$

s.t.

$$\left.\begin{aligned} AX &\leq \vec{b} \\ BX &= \vec{C} \end{aligned}\right\} G$$

$$L(x, \lambda, \mu) = m^T X - \underbrace{\lambda^T}_{\lambda \geq 0,\ \emptyset}(AX - b) - \mu^T(BX - C)$$

$$\underset{X \in G}{\boxed{\zeta(X) \leq \xi(\lambda, \mu)}} \quad \lambda \geq 0$$

= ?

Duality for General LP

$$L(x,\lambda,\mu) = m^T x - \underbrace{\lambda^T}_{\lambda \geq 0,\ \mathcal{D}}(Ax-b) - \mu^T(Bx-c)$$

$$= (m^T - \lambda^T A - \mu^T B)\,x + \lambda^T b + \mu^T c$$

no constraint

If $m^T - \lambda^T A - \mu^T B \neq 0$

$$\max_{x} L(x,\lambda,\mu) = +\infty$$

Duality for General LP

$$L(x,\lambda,\mu) = m^T x - \underbrace{\lambda^T}_{\lambda \geq 0,\ \mathcal{D}}(Ax-b) - \mu^T(Bx-c)$$

$$= (m^T - \lambda^T A - \mu^T B)\underbrace{x}_{\text{no constraint}} + \lambda^T b + \mu^T c$$

If $m^T - \lambda^T A - \mu^T B = 0$

$$\max_x L(x,\lambda,\mu) = \lambda^T b + \mu^T c$$

Duality for General LP

$$L(x,\lambda,\mu) = m^T x - \underbrace{\lambda^T}_{\lambda \geq 0,\ \mathcal{D}}(Ax-b) - \mu^T(Bx-c)$$

$$= (m^T - \lambda^T A - \mu^T B)\underbrace{x}_{\text{no constraint}} + \lambda^T b + \mu^T c$$

Hence

$$\boxed{\xi(\lambda,\mu) = \begin{cases} \lambda^T b + \mu^T c & \text{if } \lambda^T A + \mu^T B = m^T \\ +\infty & \text{if } \lambda^T A + \mu^T B \neq m^T \end{cases}}$$

Duality for General LP

$$L(x,\lambda,\mu) = m^T x - \underbrace{\lambda^T}_{\lambda \geq 0,\ \emptyset}(Ax-b) - \mu^T(Bx-c)$$

$$= (m^T - \lambda^T A - \mu^T B)\,x + \lambda^T b + \mu^T c$$

no constraint

Hence

$$\boxed{\xi(\lambda,\mu) = \begin{cases} \lambda^T b + \mu^T c & \text{if } A^T\lambda + B^T\mu = m \\ +\infty & \text{if } A^T\lambda + B^T\mu \neq m \end{cases}}$$

Duality for General LP

(P)

$$\max_{x} \; \zeta(x) = m^T x$$

s.t.

$$Ax \leq b$$

$$Bx = c$$

(D)

$$\min_{\lambda \geq 0, \mu} \; \xi(\lambda, \mu) = b^T\lambda + c^T\mu$$

s.t.

$$A^T\lambda + B^T\mu = m$$

Duality for "Traditional" LP

(P) $\max \; \zeta(x) = c^T x$

s.t. $Ax \le b$

$x \ge 0 \Leftrightarrow -x \le 0$

$$L(x, y, \lambda) = c^T x - y^T(Ax - b) - \lambda^T(-x)$$

$y, \lambda \ge 0$

$$= (c^T - y^T A + \lambda^T)x + y^T b$$

Duality for "Traditional" LP

$$L(x, y, \lambda) = c^T x - y^T(Ax - b) - \lambda^T(-x)$$

$y, \lambda \geq 0$

$$= (c^T - y^T A + \lambda^T)x + y^T b$$

$$\max_x L(x, y, \lambda) = \begin{cases} y^T b & \text{if } c^T - y^T A + \lambda^T = 0 \\ +\infty & \text{if } c^T - y^T A + \lambda^T \neq 0 \end{cases}$$

no constraint

Duality for "Traditional" LP

$$L(x, y, \lambda) = c^T x - y^T(Ax - b) - \lambda^T(-x)$$

$y, \lambda \geq 0$

$$= (c^T - y^T A + \lambda^T)x + y^T b$$

$$\max_{x} L(x, y, \lambda) = \begin{cases} y^T b & \text{if } c - A^T y + \lambda = 0 \\ +\infty & \text{if } c - A^T y + \lambda \neq 0 \end{cases}$$

no constraint

Duality for "Traditional" LP

$$(D) \quad \min_{y} \; \xi(y) = b^T y$$

$$\text{s.t.} \quad \underbrace{c - A^T y + \lambda = 0, \quad}\; y, \lambda \geq 0$$

$$\Longleftrightarrow \quad A^T y = c + \lambda, \quad y, \lambda \geq 0$$

$$\Longleftrightarrow \quad A^T y \geq c \qquad y \geq 0$$